

... the Yang-Baxter equation, and Hopf Galois structures

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- Introduction
- The Yang-Baxter equation
- Hopf Galois structures

Notation and convention

Let (G, \cdot) be a group, and $g \in G$.

- If $\psi \in \text{End}(G, \cdot)$, we write ${}^\psi g$ for the image of g under ψ .
- We denote by λ the left regular representation, and by ρ the right regular representation.
- We write $\iota: (G, \cdot) \rightarrow \text{Aut}(G, \cdot)$ for the homomorphism that sends $g \in G$ to the conjugation-by- g automorphism.
- If $\psi \in \text{End}(G, \cdot)$, we write $[g, \psi] := g \cdot {}^\psi g^{-1}$, and $[G, \psi] = \langle [g, \psi] : g \in G \rangle$.
- If (G, \cdot, \circ) is a skew brace, we denote by g^{-1} the inverse of g with respect to \cdot , and by \bar{g} the inverse of g with respect to \circ .

Theorem ([Guarnieri and Vendramin, 2017])

Let (G, \cdot) be a group. The following data are equivalent.

- An operation \circ such that (G, \cdot, \circ) is a skew brace.
- A regular subgroup $N \leq \text{Perm}(G)$ which normalises $\lambda(G)$.
- A function $\gamma: G \rightarrow \text{Aut}(G, \cdot)$ such that, for every $g, h \in G$,

$$\gamma(g \cdot \gamma(g)h) = \gamma(g)\gamma(h).$$

The function γ is called *gamma function*. Explicitly,

$$\begin{aligned}\gamma(g)h &= g^{-1} \cdot (g \circ h), \\ N &= \{\lambda(g)\gamma(g) : g \in G\}.\end{aligned}$$

Theorem ([Childs, 2019], [Caranti, 2020])

Let (G, \cdot) be a group. The following data are equivalent.

- An operation \circ such that (G, \cdot, \circ) is a bi-skew brace.
- A regular subgroup $N \leq \text{Perm}(G)$ which normalises, and is normalised by, $\lambda(G)$.
- An antihomomorphism $\gamma: (G, \cdot) \rightarrow \text{Aut}(G, \cdot)$ such that, for every $g, h \in G$,

$$\gamma(g \cdot \gamma(g)h) = \gamma(g)\gamma(h).$$

The function γ is called *bi-gamma function*.

Our setting: $\varepsilon = -1$

Let (G, \cdot) be a group, and $\psi \in \text{End}(G, \cdot)$.

Theorem ([Caranti and LS, 2021])

The following are equivalent.

- ψ satisfies ${}^\psi[[G, \psi], G] \leq Z(G, \cdot)$.
- (G, \cdot, \circ) is a bi-skew brace, for $g \circ h = g \cdot \psi g^{-1} \cdot h \cdot \psi g$.
- The function γ define by $\gamma(g) = \iota(\psi g^{-1})$ is a bi-gamma function for (G, \cdot) .

If any of these holds, then $N = \{\lambda(g)\iota(\psi g^{-1}) : g \in G\}$ is a regular subgroup of $\text{Perm}(G)$ which normalises, and is normalised by, $\lambda(G)$.

This result generalises [Koch, 2020], where the map ψ is abelian.

Main definitions

Definition ([Drinfel'd, 1992])

A *set-theoretic solution* of the Yang-Baxter equation is a couple (X, r) , where $X \neq \emptyset$ is a set, and

$$\begin{aligned} r: X \times X &\rightarrow X \times X \\ (x, y) &\mapsto (\sigma_x(y), \tau_y(x)) \end{aligned}$$

is a bijective map satisfying

$$(r \times \text{id}_X)(\text{id}_X \times r)(r \times \text{id}_X) = (\text{id}_X \times r)(r \times \text{id}_X)(\text{id}_X \times r).$$

We say that (X, r) is *non-degenerate* if, for every $x \in X$, σ_x and τ_x are bijective, and *involutive* if $r^2 = \text{id}_{X \times X}$. For us, a *solution* is a non-degenerate set-theoretic solution of the Yang-Baxter equation.

Yang-Baxter and (skew) braces

Theorem ([Rump, 2007], [Guarnieri and Vendramin, 2017])

Let (G, \cdot, \circ) be a skew brace. Then

$$r: (g, h) \mapsto (g^{-1} \cdot (g \circ h), \overline{g^{-1} \cdot (g \circ h)} \circ g \circ h)$$

is a solution for G .

The solution (G, r) is involutive if and only if (G, \cdot, \circ) is a brace, that is, if (G, \cdot) is abelian.

Definition ([Rump, 2019], [Koch and Truman, 2020a])

Let (G, \cdot, \circ) be a skew brace. The *opposite skew brace* is (G, \cdot', \circ) , where, for every $g, h \in G$, $g \cdot' h = h \cdot g$.

Given a bi-skew brace (G, \cdot, \circ) , we find (up to) four solutions for G :

$$\begin{aligned}(G, \cdot, \circ) &\rightsquigarrow (g, h) \mapsto (g^{-1} \cdot (g \circ h), \overline{g^{-1} \cdot (g \circ h)} \circ g \circ h), \\(G, \cdot', \circ) &\rightsquigarrow (g, h) \mapsto ((g \circ h) \cdot g^{-1}, \overline{(g \circ h) \cdot g^{-1}} \circ g \circ h), \\(G, \circ, \cdot) &\rightsquigarrow (g, h) \mapsto (\overline{g} \circ (g \cdot h), (\overline{g} \circ (g \cdot h))^{-1} \cdot g \cdot h), \\(G, \circ', \cdot) &\rightsquigarrow (g, h) \mapsto ((g \cdot h) \circ \overline{g}, ((g \cdot h) \circ \overline{g})^{-1} \cdot g \cdot h).\end{aligned}$$

Yang-Baxter and (bi)-gamma functions

We may rewrite the solutions so that they depend only on \cdot and γ .

Theorem ([Caranti and LS, 2021])

Let (G, \cdot) be a group, and γ be a gamma function.

Then we get (up to) two solutions:

$$(g, h) \mapsto (\gamma(g)h, \gamma(\gamma(g)h)^{-1}(\gamma(g)h^{-1} \cdot g \cdot \gamma(g)h)),$$

$$(g, h) \mapsto (\iota(g)\gamma(g)h, \gamma(\iota(g)\gamma(g)h)^{-1}g).$$

If in addition γ is a bi-gamma function, then we get (up to) other two solutions:

$$(g, h) \mapsto (\gamma(g^{-1})h, \gamma(g^{-1})h^{-1} \cdot g \cdot h),$$

$$(g, h) \mapsto (g \cdot h \cdot \gamma(h)g^{-1}, \gamma(h)g^{-1}).$$

Theorem ([Caranti and LS, 2021])

Let (G, \cdot) be a group, and $\psi \in \text{End}(G, \cdot)$. If ${}^\psi[[G, \psi], G] \leq Z(G, \cdot)$, then we get (up to) four solutions:

$$(g, h) \mapsto (\psi g^{-1} \cdot h \cdot \psi g, \psi(g^{-1} \cdot h) \cdot h^{-1} \cdot \psi g \cdot g \cdot \psi g^{-1} \cdot h \cdot \psi(h^{-1} \cdot g)),$$

$$(g, h) \mapsto (g \cdot \psi g^{-1} \cdot h \cdot \psi g \cdot g^{-1}, \psi h \cdot g \cdot \psi h^{-1}),$$

$$(g, h) \mapsto (\psi g \cdot h \cdot \psi g^{-1}, \psi g \cdot h^{-1} \cdot \psi g^{-1} \cdot g \cdot h),$$

$$(g, h) \mapsto (g \cdot h \cdot \psi h^{-1} \cdot g^{-1} \cdot \psi h, \psi h^{-1} \cdot g \cdot \psi h).$$

These coincide with the solutions found in [Koch, 2020], where ψ is abelian.

Main definition and results

Fix a finite Galois extension L/K with Galois group (G, \cdot) .

Definition

A Hopf Galois structure on L/K consists of a cocommutative K -Hopf algebra H , together with an action of H on L satisfying certain technical properties.

Theorem ([Greither and Pareigis, 1987])

The Hopf Galois structures on L/K are in bijective correspondence with the regular subgroups of $\text{Perm}(G)$ normalised by $\lambda(G)$.

The K -Hopf algebra $L[N]^G$ corresponds to the subgroup N .

Moreover, the K -sub-Hopf algebras of $L[N]^G$ are in bijective correspondence with the subgroups of N normalised by $\lambda(G)$.

Hopf Galois and gamma functions

We would like to use gamma functions to analyse Hopf Galois structures. Notice that a gamma function for (G, \cdot) yields a regular subgroup which normalises $\lambda(G)$, while we need a regular subgroup normalised by $\lambda(G)$.

Fact

If γ is a bi-gamma function for (G, \cdot) , then

$$N = \{\lambda(g)\gamma(g) : g \in G\}$$

is a regular subgroup of $\text{Perm}(G)$ which normalises, and is normalised by, $\lambda(G)$. In particular, $L[N]^G$ gives a Hopf Galois structure on L/K .

Hopf Galois structures in our setting

Let L/K be a finite Galois extension with Galois group (G, \cdot) , and $\psi \in \text{End}(G, \cdot)$ such that ${}^\psi[[G, \psi], G] \leq Z(G, \cdot)$.

Then γ , defined by $\gamma(g) = \iota({}^\psi g^{-1})$, is a bi-gamma function, and so $L[N]^G$ gives a Hopf Galois structure on L/K , where

$$N = \{\lambda(g)\iota({}^\psi g^{-1}) : g \in G\}.$$

Question

Can we determine the type of N ?

Five subgroups of N

As in [Koch, 2020], we can always find (up to) five subgroups of N normalised by $\lambda(G)$, and these correspond to five K -sub-Hopf algebras of $L[N]^G$.

For example, the λ -points and ρ -points, introduced in [Koch and Truman, 2020b]:

$$\begin{aligned}\Lambda_N &= N \cap \lambda(G) = \{\lambda(g) : g \in \ker(\gamma)\} \\ &= \{\lambda(g) : g \text{ satisfies } {}^\psi g \in Z(G, \cdot)\}, \\ P_N &= N \cap \rho(G) = \{\rho(g) : g \text{ satisfies } \gamma(g) = \iota(g^{-1})\} \\ &= \{\rho(g) : g \text{ satisfies } g \cdot {}^\psi g^{-1} \in Z(G, \cdot)\}.\end{aligned}$$

Some of the five subgroups may coincide, but we can find examples in which they are all distinct.

When we can be explicit

- If ψ is a fixed point free abelian endomorphism, then $N \cong (G, \cdot)$ ([Childs, 2013], [Koch, 2020]).
- If ψ is different from zero and idempotent, then for every $n \geq 1$, $\psi^n = \psi$, and ${}^\psi G = \{g \in G : \psi g = g\}$. We can use a version of the Fitting's Lemma for groups ([Caranti, 1985]) to deduce that $N \cong (\ker(\psi), \cdot) \times ({}^\psi G, \cdot)$.

Bibliography I



Caranti, A. (1985).

Finite p -groups of exponent p^2 in which each element commutes with its endomorphic images.

J. Algebra, 97(1):1–13.



Caranti, A. (2020).

Bi-skew braces and regular subgroups of the holomorph.

J. Algebra, 562:647–665.



Caranti, A. and LS (2021).

From endomorphisms to bi-skew braces, regular subgroups, normalising graphs, the Yang-Baxter equation, and Hopf Galois structures.

<https://arxiv.org/abs/2104.01582>.

Bibliography II



Childs, L. N. (2013).

Fixed-point free endomorphisms and Hopf Galois structures.
Proc. Amer. Math. Soc., 141(4):1255–1265.



Childs, L. N. (2019).

Bi-skew braces and Hopf Galois structures.
New York J. Math., 25:574–588.



Drinfel'd, V. G. (1992).

On some unsolved problems in quantum group theory.
In *Quantum groups (Leningrad, 1990)*, volume 1510 of *Lecture Notes in Math.*, pages 1–8. Springer, Berlin.



Greither, C. and Pareigis, B. (1987).

Hopf Galois theory for separable field extensions.
J. Algebra, 106(1):239–258.

Bibliography III



Guarnieri, L. and Vendramin, L. (2017).
Skew braces and the Yang-Baxter equation.
Math. Comp., 86(307):2519–2534.



Koch, A. (2020).
Abelian maps, bi-skew braces, and opposite pairs of
Hopf-Galois structures.
<https://arxiv.org/abs/2007.08967>.



Koch, A. and Truman, P. J. (2020a).
Opposite skew left braces and applications.
J. Algebra, 546:218–235.



Koch, A. and Truman, P. J. (2020b).
Skew left braces and isomorphism problems for Hopf-Galois
structures on Galois extensions.
<https://arxiv.org/abs/2005.05809>.



Rump, W. (2007).

Braces, radical rings, and the quantum Yang-Baxter equation.
J. Algebra, 307(1):153–170.



Rump, W. (2019).

A covering theory for non-involutive set-theoretic solutions to the Yang-Baxter equation.
J. Algebra, 520:136–170.