SCHEDULE: HOPF ALGEBRAS AND GALOIS MODULE THEORY, MAY 27–31, 2019

Half-baked ideas are welcome. This is a working conference. One of the features that everyone has enjoyed in previous conferences has been the inclusion of guiding principles, half-baked ideas, crazy conjectures. Please come with some to share.

Monday.

9:00am *Welcome*, Dave Boocker (Dean, College of Arts & Sciences)

9:15am Childs: *Hopf Galois Extensions after 50 years.* 60 minutes

10:30am Kohl: *Enumerating Hopf-Galois Structures on Dihedral Extensions.* 60 minutes

1:00pm Byott: *An introduction to integral Galois Module Structure.* 60 minutes

2:30pm Keating: *Galois scaffolds and semistable extensions.* 60 minutes

4:00pm Samways: *Enumeration of Hopf-Galois structures on cyclic field extensions.* 30 minutes

Tuesday.

9:00am Greither: *Descent theory: from abstract to concrete.* 60 minutes

10:30am Koch: *Opposite braces and their applications.* 60 minutes

1:00pm Truman: *Quotient Hopf-Galois structures and associated orders in Hopf-Galois extensions.* 60 min.

2:30pm Nejabati Zenouz: *The Yang-Baxter Equation and Hopf-Galois Theory via Skew Braces.* 60 minutes

4:00pm Fejzullahu: *Amano polynomials in positive characteristic.* 30 minutes

Evening One of the only four great Omaha Steakhouses remaining, Johnny's Cafe `http://www.johnnyscafe.com/`, as featured in the 2002 Alexander Payne movie, *About Schmidt.*

Wednesday.

9:00am Byott: *Can an insoluble Galois extension have a Hopf-Galois structure of soluble type?.* 60 minutes

10:30am Greither: *Hopf Galois theory and descent.* 60 minutes

1:00pm Underwood: *Algebraic Structure of the Canonical Non-classical Hopf Algebra.* 60 minutes

2:30pm Childs: *Galois correspondences for Hopf Galois structures arising from bi-skew braces.* 30 minutes

Thursday.

9:00am Keating: *Stable and semistable Hopf-Galois extensions.* 60 minutes

10:30am Koch: *Isomorphism problems on Hopf algebras and braces.* 60 minutes

1:00pm Kohl: *Multiple Holomorphs and Isomorphism Classes of Hopf-Galois Structures on Dihedral Extensions.* 60 minutes

2:30pm Truman: *Hopf-Galois module structure of tamely ramified radical extensions of prime degree.* 30 minutes

Friday.

9:00am Underwood: *Hopf Orders in KC_{p^3}.* 60 minutes

10:30am Elder: *Sharp lower bounds on ramification breaks in non-abelian extensions of degree p^3.* 60 minutes

Evening Pool party at the Elder-berry Residence: 5624 Leavenworth St.

Abstracts

Nigel Byott, University of Exeter.

An introduction to integral Galois Module Structure. 60 minutes

Abstract: I will give an introductory survey of some results on the Galois module structure of rings of integers in the context of number fields and p-adic fields. The emphasis will be on foundational results and concrete examples.

Can an insoluble Galois extension have a Hopf-Galois structure of soluble type? 60 minutes

Abstract: A recent preprint of Tsang and Chao gives some evidence in support of a negative answer to this question. I will describe work in progress which strengthens some of their results. In particular, I will outline a proof (depending on the Classification of Finite Simple Groups) that, in a minimal example, the nonabelian composition factors of the Galois group must be of the form $\mathrm{PSL}_n(q)$ for $n = 2$ or 3, with strong restrictions on the prime power q.

Lindsay Childs, University of Albany.

Hopf Galois Extensions after 50 years. 60 minutes

Abstract: In 1969 Chase and Sweedler published their paper, "Hopf algebras and Galois theory". They wrote: "The motivation for this work was a hope that results of this type should shed some light on inseparable extensions of fields and ramified extensions of rings." In this talk we define a Hopf Galois extension, discuss local Galois module theory and why Hopf Galois extensions are of interest there, and then focus on the question: given a Galois extension of fields with Galois group G, what types of Hopf Galois structures are possible on the Galois extension. The adventure involves ideas involving regular subgroups of permutation groups, radical algebras and, quite recently, new algebraic objects, called left skew braces, that arose independently from considerations in mathematical physics.

Galois correspondences for Hopf Galois structures arising from bi-skew braces. 30 minutes

Abstract: A bi-skew brace is a set $(B, \circ, \star)$ with two group structures so that with either operation defining the additive group, B is a skew left brace. An ordered pair (G, N) of finite groups of equal order will be called realizable if there exists a Galois extension L/K with Galois group G with a Hopf Galois structure of type N. Given a bi-skew brace $(B, \circ, \star)$, then both $((B, \circ), (B, \star))$ and $((B, \star), (B, \circ))$ are realizable. Two general classes of bi-skew braces arise from semidirect products of groups, and from radical algebras A with $A^3 = 0$. For each of the two classes, we characterize the images of the Galois correspondences for both Hopf Galois structures in terms of subobjects of the bi-skew brace, and see what those subobjects look like for some examples.

Griff Elder, University of Nebraska at Omaha.

Sharp lower bounds on ramification breaks in non-abelian extensions of degree p^3. 60 minutes

Abstract: Let L/K be a totally ramified Galois extension of local number fields having residue characteristic p. When the Galois group $G = \mathrm{Gal}(L/K)$ is abelian, the ramification filtration of the Galois group is fairly well understood. (See work of Maus and Miki.) However when the Galois group is nonabelian, the Hasse-Arf Theorem no longer applies, the upper ramification breaks no longer need be integers, and as illustrated in work with J. Hooper on quaternion extensions, the ramification breaks can exhibit some interesting features. In this talk, we will discuss the two nonabelian groups of order p^3, $p > 2$. To avoid the embedding problem, we will assume $\mathrm{char}(K) = p$.

Endrit Fejzullahu, University of Florida.

Amano polynomials in positive characteristic. 30 minutes

Abstract: Let K be a local field of characteristic p and let L/K be a totally ramified extension of degree p^k having two indices of inseparability. We show that there exists a uniformizer of L whose minimum polynomial over K has only three nonzero terms.

Cornelius Greither, Universität der Bundeswehr München.

Descent theory: from abstract to concrete. 60 minutes

Abstract: Here I will try to begin with the fairly general setup of Grothendieck (but scaled down from schemes to rings). Then I plan to take up the various types of descent formalisms, and explain how one arises from another: faithfully flat, Zariski, faithfully projective, Galois descent. It doesn't make sense to cover all variants at the same level of detail; the opposite extremes are Galois descent, which will be covered carefully because in the end it matters most for us, and radical descent, which will be given short shrift. It is my intention to include quite a few examples all along.

Hopf Galois theory and descent. 60 minutes

Abstract: Here I would like to discuss how descent theory was applied in the last years by various people to obtain information on the isomorphism class of Hopf algebras that occur in Hopf Galois theory. It appears that this is still an ongoing story. There will be a strong focus on particular cases, not only for commendable expository reasons, but because it seems that general results, though a few are available, do not yet dominate the picture. The "various people" alluded to above are A. Koch, T. Kohl, S. Taylor, P. Truman, R. Underwood, and the speaker.

Kevin Keating, University of Florida.

Galois scaffolds and semistable extensions. 60 minutes

Abstract: Let K be a local field and let L/K be a totally ramified Galois extension of degree p^n. Being semistable and possessing a Galois scaffold are two conditions which facilitate the computation of the additive Galois module structure of L/K. In this talk I will consider the relation between semistable extensions and Galois scaffolds. The main result is that L/K is semistable if and only if L/K has a Galois scaffold with precision 1.

Stable and semistable Hopf-Galois extensions. 60 minutes

Abstract: Let K be a local field and let H be a Hopf algebra over K. Let L/K be a totally ramified H-Galois extension of degree p^n. In this talk I will formulate what it means for L/K to be H-stable or H-semistable. I will then consider the problem of determining to what extent H-stable and H-semistable extensions have properties analogous to those proved by Bondarko for stable and semistable Galois extensions. This is a work in progress; many questions remain unanswered or unasked.

Alan Koch, Agnes Scott College.

Opposite braces and their applications. 60 minutes

Abstract: Let L/K be a finite Galois extension, and let $G = \mathrm{Gal}(L/K)$. Let $N \leq \mathrm{Perm}(G)$ be a regular subgroup normalized by $\lambda(G) \leq \mathrm{Perm}(G)$, the group of left regular representations of G on itself: such an N gives rise to a Hopf-Galois structure on L/K. It is well known that $N' = \mathrm{Cent}_{\mathrm{Perm}(G)}(N)$ is also a regular subgroup normalized by $\lambda(G)$, hence N' also gives rise to a Hopf-Galois structure on L/K. Associated to (G, N) is a skew left brace $\mathfrak{B} = (B, \cdot, \circ)$, an object developed to give non-degenerate set-theoretic solutions to the Yang-Baxter equation. Of course, (G, N') must also correspond to a skew left brace, say $\mathfrak{B}'$, which we call the *opposite* brace to $\mathfrak{B}$. We construct $\mathfrak{B}'$ and describe some of its basic properties. As applications, we discuss the corresponding solutions to the Yang-Baxter equation, and illustrate how the opposite brace directly connects the left ideal structure of $\mathfrak{B}$ to the collection of subfields obtained in the Hopf-Galois correspondence.

Isomorphism problems on Hopf algebras and braces. 60 minutes

Abstract: Let L/K be Galois. Suppose L/K is Hopf Galois for a collection of cocommutative Hopf algebras $H_1, H_2, \cdots, H_n$. Associated to H_i, $i = 1, 2, \ldots, n$ is a regular subgroup N_i of $\mathrm{Perm}(G)$, and this corresponds to a skew left brace $\mathfrak{B}_i$. We will investigate connections between isomorphisms of H_i and H_j (as K-Hopf algebras) and isomorphisms between their respective braces $\mathfrak{B}_i$, $\mathfrak{B}_j$. We pay particular attention to the case where $\mathfrak{B}_j = \mathfrak{B}'_i$, that is, the braces are opposite to each other.

Tim Kohl, Boston University.

Enumerating Hopf-Galois Structures on Dihedral Extensions. 60 minutes

Abstract: For K/k a Galois extension where $G = Gal(K/k) \cong D_n$ the n-th dihedral group, we enumerate $R(G, [G])$ the regular permutation groups $N \leq B = Perm(G)$ (where $N \cong G$) which give rise to Hopf-Galois structures on K/k as detailed in the Greither-Pareigis theory. We use the fact that the possible maximal blocks of any $N \in R(G, [G])$, and concordantly $Norm_B(N)$, are determined by those of $\lambda(G)$. The distribution of the N amongst the three possible systems of maximal blocks as well as how the cardinality of the multiple holomorph $|T(D_n)| = [Norm_{Perm(D_n)}(Hol(D_n)) : Hol(D_n)]$ impacts the enumeration is given. We will also see the sharp distinction between the case where n is odd versus when n is even.

Multiple Holomorphs and Isomorphism Classes of Hopf-Galois Structures on Dihedral Extensions. 60 minutes

Abstract: For Galois extensions K/k for which $G = Gal(K/k) \cong D_n$, we consider the Hopf-Galois structures on K/k for which $H = (K[N])^G$ where $G \cong N \cong D_n$, for n odd. In particular we show how these are in one-to-one correspondence with $|T(D_n)| = [Norm_{Perm(D_n)}(Hol(D_n)) : Hol(D_n)]$, and that this correspondence also implies that none of the Hopf-algebras which arise are isomorphic as Hopf-algebras. We also consider, in contrast, the isomorphism class(es) of these H as k-algebras. Part of this analysis is made possible by the usage of a standard basis for these H as fixed rings. This basis also lends itself to an understanding of the ring structure of these H, keyed to normal bases of K/k, as well as other intermediate fields therein.

Kayvan Nejabati Zenouz, Oxford Brookes University.

The Yang-Baxter Equation and Hopf-Galois Theory via Skew Braces. 60 minutes

Abstract: The Yang-Baxter equation is a matrix equation for the linear automorphisms of the tensor product of a vector space with itself. This equation plays a central role in quantum group theory and appears in many areas of mathematical physics. On the other hand, Hopf-Galois structures on Galois extensions of number fields are of great interest in Galois module theory as they reveal information about the rings of integers of these extensions. The classification of both the solutions Yang-Baxter equation and Hopf-Galois structures remain among important topics of research. In this talk we will explain how Hopf-Galois theory and the Yang-Baxter equation found to be related via algebraic objects known as skew braces. Then we will review our results on classification of skew braces of order p^3, their automorphism groups, and their connection to Hopf-Galois structures.

George Samways, University of Exeter.

Enumeration of Hopf-Galois structures on cyclic field extensions. 30 minutes

Abstract: We seek to enumerate Hopf-Galois structures on a cyclic Galois extension L/K of degree n. By Greither and Pareigis, such structures may be associated with some group G of order n. In particular, such a group must be supersolvable, and every p-Sylow subgroup is cyclic for p an odd prime. In the case where the 2-Sylow subgroup of G is also cyclic, we provide an enumeration of the Hopf-Galois structures on L/K with type G.

Paul Truman, Keele University.

Quotient Hopf-Galois structures and associated orders in Hopf-Galois extensions. 60 minutes

Abstract: Let L/K be a finite Galois extension of fields with Galois group G. By the Greither-Pareigis classification, each Hopf algebra giving a Hopf-Galois structure on L/K has the form $L[N]^G$ for some regular subgroup N of $\mathrm{Perm}(G)$ that is normalized by $\lambda(G)$, the image of G under the left regular representation. If P is a subgroup of N that is also normalized by $\lambda(G)$ then $L[P]^G$ is a Hopf subalgebra of $L[N]^G$, which gives rise to a fixed field L^P and a Hopf-Galois structure on L/L^P. In recent work with Koch, Kohl, and Underwood, we showed that if in addition P is a normal subgroup of N then P gives rise to a "quotient" Hopf-Galois structure on L^P/K. In this talk we study some applications of this construction to questions of integral Hopf-Galois module structure in extensions of local or global fields.

Hopf-Galois module structure of tamely ramified radical extensions of prime degree. 30 minutes

Abstract: Let K be a number field and let L/K be a tamely ramified radical extension of prime degree p. If K contains a primitive p^{th} root of unity then L/K is a cyclic Kummer extension: in this case the group algebra $K[G]$ (with $G = \mathrm{Gal}(L/K)$) gives the unique Hopf-Galois structure on L/K, the ring of algebraic integers $\mathfrak{O}_L$ is a locally free $\mathfrak{O}_K[G]$-module by Noether's theorem, and Gómez Ayala has determined a criterion for $\mathfrak{O}_L$ to be a free $\mathfrak{O}_K[G]$-module. If K does not contain a primitive p^{th} root of unity then L/K is a separable, but non-normal, extension, which again admits a unique Hopf-Galois structure. Under the assumption that p is unramified in K, we show that $\mathfrak{O}_L$ is locally free over its associated order in this Hopf-Galois structure and determine a criterion for it to be free. We find that the conditions that appear in this criterion are identical to those appearing in Góomez Ayalas criterion for the normal case.

Rob Underwood, Auburn University at Montgomery.

Algebraic Structure of the Canonical Non-classical Hopf Algebra. 60 minutes

Abstract: Let L/K be a Galois extension with non-abelian group G. Then L/K admits both a classical and canonical non-classical Hopf-Galois structure via the Hopf algebras $K[G]$ and H_λ, respectively. By a theorem of C. Greither, $K[G] \cong H_\lambda$ as K-algebras. Various proofs of Greither's result have been found for specific Galois groups. For instance, S. Taylor and P. J. Truman have shown that $K[G] \cong H_\lambda$ when G is the quaternion group Q_8, and U. has shown that $K[G] \cong H_\lambda$ for the cases $G = D_4$ and $G = D_3$. We examine the D_3 case in detail in an attempt to find explicit formulas for the matrix units in H_λ.

Hopf Orders in KC_{p^3}. 60 minutes

Abstract: Let p be a prime number and let $n \geq 1$. Let K be a finite extension of the p-adic rationals $\mathbb{Q}_p$ containing ζ_n, a primitive p^nth root of unity. Let C_{p^n} denote the cyclic group of order p^n and let C_p^n denote the elementary abelian group of order p^n. In the case $n = 3$, U. has constructed a collection of Hopf orders in KC_{p^3}, some of which are realizable. We relate these Hopf orders to the realizable Hopf orders in KC_p^3 obtained by N. P. Byott and G. G. Elder.