The Yang-Baxter Equation and Hopf-Galois Theory via Skew Braces

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Braces and Hopf-Galois Theory
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Outline

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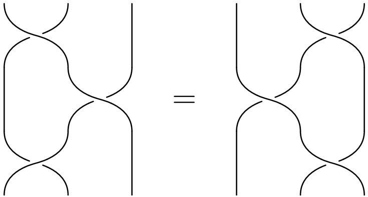

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holds. The solution (X, r) is called **non-degenerate** if $f_x, g_x \in \mathrm{Perm}(X)$ for all $x \in X$ and **involutive** if $r^2 = \mathrm{id}$.

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3. For any group structure on X the map
$$r(x,y) = (y, yxy^{-1}).$$
4. If $(R, +, \cdot)$ is a radical ring with circle operation $a \circ b = a + ab + b$ then $r(x,y) = (xy + y, (xy+y)^{\circ -1} \circ x \circ y)$.

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Interesting for ring theorists: $0 = 1$.

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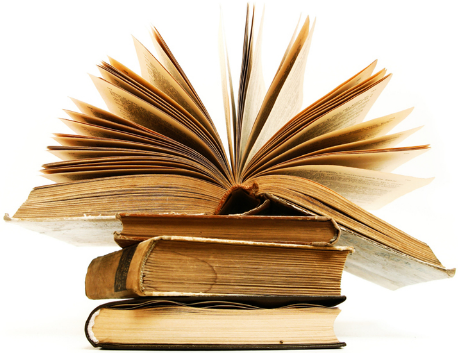

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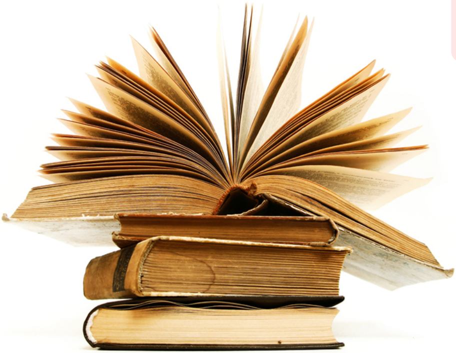

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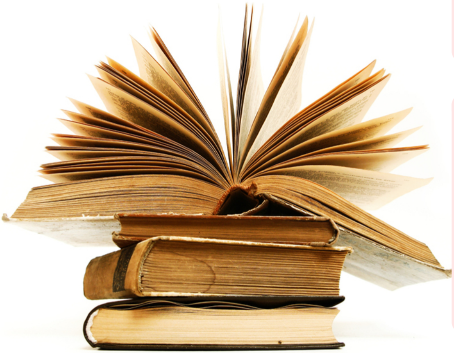

They provide *non-degenerate* **set-theoretic solutions of the Yang-Baxter equation**.

Their connection to **ring theory** and **Hopf-Galois structures** was studied by Bachiller, Byott, Smoktunowicz, and Vendramin.

Skew Braces and the YBE

Theorem (L. Guarnieri and L. Vendramin)

Let $(B, \oplus, \odot)$ be a skew brace. Then the map

$$\begin{aligned} r_B : B \times B &\longrightarrow B \times B \\ (a, b) &\longmapsto \big(\ominus a \oplus (a \odot b), (\ominus a \oplus (a \odot b))^{-1} \odot a \odot b\big) \end{aligned}$$

is a non-degenerate set-theoretic solution of the YBE, which is involutive if and only if $(B, \oplus, \odot)$ is a brace.

Relation to Rings

- Given a skew brace $(B, \oplus, \odot)$ define

$$a \otimes b = \ominus a \oplus (a \odot b) \ominus b.$$

Cedo, Konovalov, Vendramin, Smoktunowicz (2018) study $(B, \oplus, \otimes)$ using ring theoretic methods.

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- However, if B is a **two-sided brace**, then $(B, \oplus, \otimes)$ is a **radical ring**, Rump (2007).
- Conversely, if $(B, \oplus, \otimes)$ is a **radical ring**, then $(B, \oplus, \circ)$, where

$$a \circ b = a \oplus a \otimes b \oplus b$$

  is a **two-sided brace**, Rump (2007).

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Studying **rings of integers** of extensions of **number fields**.

Hopf-Galois Structures: Motivations

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- Then $\mathcal{O}_L$ is also a module over $\mathcal{O}_K[G]$.
- Can $\mathcal{O}_L$ be free over $\mathcal{O}_K[G]$?

... No in general.

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A **Hopf-Galois structure** on L/K consists of a finite dimensional cocommutative K-*Hopf algebra* H together with an action on L such that the R-module homomorphism

$$\begin{aligned} j : & L \otimes_K H \longrightarrow \mathrm{End}_K(L) \\ & s \otimes h \longmapsto (t \longmapsto sh(t)) \text{ for } s, t \in L, \ h \in H \end{aligned}$$

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The **group algebra** $K[G]$ endows L/K with the classical Hopf-Galois structure.

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- Can $\mathcal{O}_L$ be free over $\mathfrak{A}_H$?
- How to find Hopf-Galois structures?

Hopf-Galois Structures: A Theorem of Greither and Pareigis

Theorem (Greither and Pareigis)

Hopf-Galois structures on L/K correspond bijectively to regular subgroups of $\mathrm{Perm}(G)$ *which are normalised by the image of* G*, as left translations, inside* $\mathrm{Perm}(G)$.

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Every K-Hopf algebra which endows L/K with a Hopf-Galois structure is of the form $L[N]^G$ for some regular subgroup $N \subseteq \mathrm{Perm}(G)$ normalised by the left translations.

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Instead of working with groups of permutations, work with *holomorphs*.

Theorem (Byott 1996)

Let G and N be finite groups. There exists a bijection between the sets

$$\mathcal{N} = \{\alpha : N \hookrightarrow \mathrm{Perm}(G) \mid \alpha(N) \textit{ is regular and normalised by } G\}$$

$$\mathcal{G} = \{\beta : G \hookrightarrow \mathrm{Hol}(N) \mid \beta(G) \textit{ is regular}\},$$

where $\mathrm{Hol}(N) = N \rtimes \mathrm{Aut}(N)$.

Hopf-Galois Structures: Byott's Translation

Enumerating Hopf-Galois Structures (Byott)

Using Byott's translation one can show that

$$\sharp\text{HGS on } L/K \text{ of type } N = \frac{|\mathrm{Aut}\,(G)|}{|\mathrm{Aut}\,(N)|}\left|\{H \subseteq \mathrm{Hol}(N) \text{ regular with } H \cong G\}\right|.$$

Hopf-Galois Structures: Some Results

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- Crespo and Salguero extensions of degree $C_{p^n} \rtimes C_D$, Samways cyclic extensions, and Tsang S_n-extensions.

Hopf-Galois Structures of Order p^3 for $p > 3$

Theorem 1 [cf. NZ18, Jan 2018]

The number of Hopf-Galois structures on L/K of type N, $e(G, N)$, is given by

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$e(G,N)$	C_{p^3}	$C_{p^2} \times C_p$	C_p^3	$C_p^2 \rtimes C_p$	$C_{p^2} \rtimes C_p$
C_{p^3}	p^2	-	-	-	-
$C_{p^2} \times C_p$	-	$(2p-1)p^2$	-	-	$(2p-1)(p-1)p^2$
C_p^3	-	-	$(p^4+p^3-1)p^2$	$(p^3-1)(p^2+p-1)p^2$	-
$C_p^2 \rtimes C_p$	-	-	$(p^2+p-1)p^2$	$(2p^3-3p+1)p^2$	-
$C_{p^2} \rtimes C_p$	-	$(2p-1)p^2$	-	-	$(2p-1)(p-1)p^2$

Column $C_p^2 \rtimes C_p$ J. Algebra [cf. NZ19, Apr 2019]. Cases $p = 2, 3$ are also treated in PhD thesis.

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$e(G,N)$	C_{p^3}	$C_{p^2} \times C_p$	C_p^3	$C_p^2 \rtimes C_p$	$C_{p^2} \rtimes C_p$
C_{p^3}	p^2	-	-	-	-
$C_{p^2} \times C_p$	-	$(2p-1)p^2$	-	-	$(2p-1)(p-1)p^2$
C_p^3	-	-	$(p^4+p^3-1)p^2$	$(p^3-1)(p^2+p-1)p^2$	-
$C_p^2 \rtimes C_p$	-	-	$(p^2+p-1)p^2$	$(2p^3-3p+1)p^2$	-
$C_{p^2} \rtimes C_p$	-	$(2p-1)p^2$	-	-	$(2p-1)(p-1)p^2$

Column $C_p^2 \rtimes C_p$ J. Algebra [cf. NZ19, Apr 2019]. Cases $p = 2, 3$ are also treated in PhD thesis.

Remark

Note $p^2 \mid e(G,N)$ and

$$|\mathrm{Aut}(N)|\, e(G,N) = |\mathrm{Aut}(G)|\, e(N,G).$$

Corollaries

Denote by

$$e(G) = \sum_N e(G,N) \text{ and } \overline{e}(N) = \sum_G e(G,N).$$

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Then we have

G	$e(G)$	$\overline{e}(G)$
C_{p^3}	p^2	p^2
$C_{p^2} \times C_p$	$(2p-1)p^3$	$2(2p-1)p^2$
C_p^3	$(p^4+2p^3-p-1)p^3$	$(p^4+p^3+p^2+p-2)p^2$
$C_p^2 \rtimes C_p$	$(2p^2+p-2)p^3$	$(p^5+p^4+p^3-p^2-4p+2)p^2$
$C_{p^2} \rtimes C_p$	$(2p-1)p^3$	$2(2p-1)(p-1)p^2$
Total		$(p^5+2p^4+2p^3+4p^2-5p+1)p^2$

Hopf-Galois Structures and Skew Braces

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Skew braces parametrise Hopf-Galois structures.

$$\left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of } G\text{-skew braces,}\\ \text{i.e., with } (B,\odot)\cong G\end{array}\right\} \overset{\text{bij}}{\leftrightsquigarrow} \left\{\begin{array}{c}\text{classes of certain regular}\\ \text{subgroups of Perm(G) under}\\ \text{conjugation by elements of}\\ \mathrm{Aut}(G)\end{array}\right\}$$

- Suppose $(B, \oplus, \odot)$ is a skew brace.

From Skew Braces to Hopf-Galois Structures

- Suppose $(B, \oplus, \odot)$ is a skew brace.
- Then $(B, \oplus)$ acts on $(B, \odot)$ and we find

$$\begin{aligned} d : (B, \oplus) &\longrightarrow \operatorname{Perm}(B, \odot) \\ a &\longmapsto (d_a : \ b \longmapsto a \oplus b) \end{aligned}$$

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- Isomorphic skew braces correspond to conjugate regular subgroups.

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- Suppose H endows L/K with a Hopf-Galois structure.

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- N is normalised by the left translations implies that $(B, \oplus, \odot)$ is a skew brace of type N corresponding to H.

Skew Braces and Hopf-Galois Structures Correspondence

$$\left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of } G\text{-skew braces,}\\ \text{i.e., with } (B,\odot)\cong G\end{array}\right\} \overset{\text{bij}}{\leftrightsquigarrow} \left\{\begin{array}{c}\text{classes of Hopf-Galois structures}\\ \text{on } L/K \text{ under } L[N_1]^G \sim L[N_2]^G\\ \text{if } N_2=\alpha N_1\alpha^{-1} \text{ for some}\\ \alpha\in\mathrm{Aut}(G)\end{array}\right\}$$

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i.e., if $(B,\oplus,\odot)$ is a skew brace of type, then we get the

following Hopf-Galois structures on L/K

$$\left\{L[\alpha\,(\mathrm{Im}\,d)\,\alpha^{-1}]^{(B,\odot)} \mid \alpha\in\mathrm{Aut}\,(B,\odot)\right\}.$$

Upshot: Automorphism Groups of Skew Braces

Automorphism Groups [cf. NZ19, Apr 2019, Corollary 2.3]

In particular, if $f : (B, \oplus, \odot) \longrightarrow (B, \oplus, \odot)$ is an automorphism,

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\end{tikzcd}$$

using this observation we find

$$\mathrm{Aut}_{\mathcal{B}r}\,(B, \oplus, \odot) \cong \left\{\alpha \in \mathrm{Aut}\,(B, \odot) \mid \alpha\,(\mathrm{Im}\,d)\,\alpha^{-1} \subseteq \mathrm{Im}\,d\right\}.$$

Classification of Hopf-Galois Structures and Skew Braces: Theoretical

Classifying Skew Braces

To find the non-isomorphic G-skew braces of type N classify elements of the set

$$\mathcal{S}(G, N) = \{H \subseteq \operatorname{Perm}(G) \mid H \text{ is regular, NLT, } H \cong N\},$$

and extract a maximal subset whose elements are not conjugate by any element of $\operatorname{Aut}(G)$.

Classification of Hopf-Galois Structures and Skew Braces: Theoretical

Hopf-Galois Structures Parametrised by Skew Braces [cf. NZ19, Corollary 2.4]

Denote by B_G^N the isomorphism class of a G-skew brace of type N given by $(B, \oplus, \odot)$.

Classification of Hopf-Galois Structures and Skew Braces: Theoretical

Hopf-Galois Structures Parametrised by Skew Braces [cf. NZ19, Corollary 2.4]

Denote by B_G^N the isomorphism class of a G-skew brace of type N given by $(B, \oplus, \odot)$. Then the number of Hopf-Galois structures on L/K of type N is given by

$$e(G, N) = \sum_{B_G^N} \frac{|\mathrm{Aut}\,(G)|}{|\mathrm{Aut}_{\mathcal{B}r}\,(B_G^N)|}.$$

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Again we would like to work with **holomorphs** instead of the **permutation groups**.

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For a skew brace $(B, \oplus, \odot)$ consider the action of $(B, \odot)$ on $(B, \oplus)$ by $(a, b) \longmapsto a \odot b$. This yeilds to a map

$$\begin{aligned} m : (B, \odot) &\longrightarrow \mathrm{Hol}\,(B, \oplus) \\ a &\longmapsto (m_a : b \longmapsto a \odot b) \end{aligned}$$

which is a regular embedding.

Skew Braces and Regular Subgroups of Holomorph Correspondence

Bachiller, Byott, Vendramin:

$$\left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of skew braces of}\\ \text{type } N\text{, i.e., with}\\ (B,\oplus)\cong N\end{array}\right\} \overset{\text{bij}}{\leftrightsquigarrow} \left\{\begin{array}{c}\text{classes of regular subgroup of}\\ \mathrm{Hol}(N)\text{ under } H_1\sim H_2 \text{ if}\\ H_2=\alpha H_1\alpha^{-1}\text{ for some}\\ \alpha\in\mathrm{Aut}(N)\end{array}\right\}$$

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Another Characterisation of Automorphism Group [cf. NZ18, Jan 2018, Theorem 2.3.8, p 29]

We find

$$\mathrm{Aut}_{\mathcal{B}r}\left(B,\oplus,\odot\right)\cong\left\{\alpha\in\mathrm{Aut}\left(B,\oplus\right)\mid\alpha\left(\mathrm{Im}\,m\right)\alpha^{-1}\subseteq\mathrm{Im}\,m\right\}.$$

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- Timur Nasybullov (2018), **two-sided skew braces**.
- Koch and Truman (2019), **opposite braces** and **isomorphism correspondence**.

Skew Braces of Order p^3 for $p > 3$

Theorem 2 [cf. NZ18, Jan 2018]

The number of G-skew braces of type N, $\widetilde{e}(G, N)$, is given by

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$\widetilde{e}(G,N)$	C_{p^3}	$C_{p^2} \times C_p$	C_p^3	$C_p^2 \rtimes C_p$	$C_{p^2} \rtimes C_p$
C_{p^3}	3	-	-	-	-
$C_{p^2} \times C_p$	-	9	-	-	$4p+1$
C_p^3	-	-	5	$2p+1$	-
$C_p^2 \rtimes C_p$	-	-	$2p+1$	$2p^2-p+3$	-
$C_{p^2} \rtimes C_p$	-	$4p+1$	-	-	$4p^2-3p-1$

Column $C_p^2 \rtimes C_p$ and automorphism groups [cf. NZ19, Apr 2019].

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C_p^3	-	-	5	$2p+1$	-
$C_p^2 \rtimes C_p$	-	-	$2p+1$	$2p^2 - p + 3$	-
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Remark

Note

$$\widetilde{e}(G, N) = \widetilde{e}(N, G).$$

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Then we have

G	$\widetilde{e}(G)$
C_{p^3}	3
$C_{p^2} \times C_p$	$4p + 10$
C_p^3	$2p + 6$
$C_p^2 \rtimes C_p$	$2p^2 + p + 4$
$C_{p^2} \rtimes C_p$	$4p^2 + p$
Total	$6p^2 + 8p + 23$

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$$\mathrm{Aut}(C_{p^3}) \cong C_{p^2} \times C_{p-1}, \; \mathrm{Aut}(C_p^3) \cong \mathrm{GL}_3(\mathbb{F}_p),$$

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- Classify regular subgroups of $\mathrm{Hol}(N)$ according to the size of their image under the natural projection

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$$\mathrm{Hol}(N) \longrightarrow \mathrm{Aut}(N).$$

- To find **skew braces** study conjugation formula by elements of $\mathrm{Aut}(N)$ inside $\mathrm{Hol}(N)$.

Strategy for the Proofs

- Organise the regular subgroups of $H \subset \mathrm{Hol}(N)$ according to the size of their image under the projection

$$\Theta : \mathrm{Hol}(N) \longrightarrow \mathrm{Aut}(N) \qquad \eta\alpha \longmapsto \alpha.$$

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- Consider subgroups of $\mathrm{Hol}(N)$ of the form

$$H = \langle \eta_1, ..., \eta_r, v_1\alpha_1, ..., v_s\alpha_s \rangle \subseteq \mathrm{Hol}(N).$$

Strategy for the Proofs

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- For H to act freely on N it is necessary that for every word $W(\alpha_1, ..., \alpha_s) \neq 1$ in H_2 we require

$$W(u_1(v_1\alpha_1)w_1, ..., u_s(v_s\alpha_s)w_s)W(\alpha_1, ..., \alpha_s)^{-1} \notin H_1$$

  for all $u_i, w_i \in H_1$.

Hopf-Galois Structures of Heisenberg Type

Heisenberg Group

$$C_p^2 \rtimes C_p = \langle \rho, \sigma, \tau \mid \rho^p = \sigma^p = \tau^p = 1,\ \sigma\rho = \rho\sigma,\ \tau\rho = \rho\tau,\ \tau\sigma = \rho\sigma\tau \rangle$$

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Let us denote by

$$\alpha_1 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \alpha_2 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},\ \alpha_3 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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We have $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \cong C_p^2 \rtimes C_p$ is one of the $p+1$ Sylow p-subgroups of

$$\mathrm{Aut}(C_p^2 \rtimes C_p) \cong C_p^2 \rtimes \mathrm{GL}_2(\mathbb{F}_p).$$

Nonabelian:

$$\left\langle \rho, \tau, \sigma\alpha_1^b \right\rangle, \left\langle \rho, \tau, \sigma\alpha_1^a\alpha_2^b \right\rangle, \langle \rho, \tau, \sigma\alpha_1^a\alpha_3^c \rangle, \left\langle \rho, \tau, \sigma\alpha_1^a\alpha_2^b\alpha_3^c \right\rangle$$

$$\text{for } a = 0, ..., p-1, \ b, c = 1, ..., p-1, \text{with } c \neq 1,$$

$$\left\langle \rho, \sigma\tau^d, \tau\alpha_1^b \right\rangle, \left\langle \rho, \sigma\tau^d, \tau\alpha_1^a\alpha_3^c \right\rangle$$

for $a, d = 0, ..., p-1, \ b, c = 1, ..., p-1$ with $b \neq p-1, \ a+cd+1 \not\equiv 0 \bmod p$.

Hopf-Galois Structures of Heisenberg Type (p)

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$$\left\langle \rho, \tau, \sigma\alpha_1^b \right\rangle, \left\langle \rho, \tau, \sigma\alpha_1^a\alpha_2^b \right\rangle, \langle \rho, \tau, \sigma\alpha_1^a\alpha_3^c \rangle, \left\langle \rho, \tau, \sigma\alpha_1^a\alpha_2^b\alpha_3^c \right\rangle$$

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We shall multiply by $p+1$ appropriately wherever a subgroup involves α_2.

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Skew Braces:

$$\langle \rho, \tau, \sigma\alpha_3 \rangle, \langle \rho, \tau, \sigma\alpha_2\alpha_3 \rangle \cong C_p^3,\ \langle \rho, \tau, \sigma\alpha_1 \rangle, \langle \rho, \tau, \sigma\alpha_2 \rangle,$$

$$\langle \rho, \tau, \sigma\alpha_3^c \rangle, \langle \rho, \tau, \sigma\alpha_2\alpha_3^c \rangle \cong M_1 \text{ for } c = 2, ..., p-1.$$

Nonabelian:

$\langle \rho, u\alpha_1, v\alpha_3 \rangle$ for $A = \left(\begin{smallmatrix} u_2 & v_2 \\ u_3 & v_3 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{F}_p)$ with $v_2 - u_3 - \det(A) \not\equiv 0 \bmod p$,
$\langle \rho, \tau^{x_3}\alpha_1, y\alpha_2\alpha_3^a \rangle$ for $a, y_3 = 0, ..., p-1$, $y_2, x_3 = 1, ..., p-1$
with $y_2 - ax_3 + x_3 y_2 \not\equiv 0 \bmod p$,

Hopf-Galois Structures of Heisenberg Type (p^2)

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Skew braces:

$$\langle \rho, \sigma\alpha_1, \sigma^{u_3}\tau^{u_4}\alpha_3 \rangle, \langle \rho, \tau^{-u_5}\alpha_1, \sigma^{u_5}\alpha_3 \rangle, \langle \rho, \tau^{x_3}\alpha_1, \sigma\alpha_2\alpha_3^a \rangle \cong M_1,$$

$$\langle \rho, \sigma\alpha_1, \sigma^{u_2}\tau^{u_2}\alpha_3 \rangle, \langle \rho, \tau^{-2}\alpha_1, \sigma^2\alpha_3 \rangle, \left\langle \rho, \tau^{x_3}\alpha_1, \sigma\alpha_2\alpha_3^{(1+x_3)x_3^{-1}} \right\rangle \cong C_p^3 \text{ for}$$

$$a, u_3 = 0, ..., p-1,\ u_2, u_4, u_5, x_3, = 1, ..., p-1$$

$$\text{with } u_5 \neq 2,\ u_3 - u_4,\ ax_3 - (1+x_3) \not\equiv 0 \bmod p.$$

Abelian:

$$\left\langle \rho^{u_1}\tau^{-2}\alpha_1, \rho^{v_1}\tau^{1-u_1}\alpha_2, \rho^{w_1}\sigma^2\tau^{w_3}\alpha_3 \right\rangle \cong M_1$$

for $u_1, v_1, w_1, w_3 = 0, ..., p-1$ with $v_1 + \dfrac{1}{2}u_1(1-u_1) \not\equiv 0 \bmod p$,

Skew braces:

$$\left\langle \tau^{-2}\alpha_1, \rho^s\tau\alpha_2, \sigma^2\tau^{t_3}\alpha_3 \right\rangle \cong M_1 \text{ for } t_3 = 0, 1,\ \ s = 1, \delta,$$

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Example

Let $p > 2$, $n > 1$, and $C_{p^n} = \left\langle \sigma \mid \sigma^{p^n} = 1 \right\rangle$.

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We also have

$$\mathrm{Aut}_{\mathcal{B}r}\left(\left\langle \sigma \beta^{p^m} \right\rangle\right) = \left\langle \beta^{p^{n-m-2}} \right\rangle \text{ for } m = 0, \ldots, n-2.$$

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Skew Braces of Semi-direct Product Type

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Proposition 4.6.12 [cf. NZ18, Jan 2018, p. 130]

Let P and Q be groups. Suppose $\alpha, \beta : Q \longrightarrow \mathrm{Aut}(P)$ are group homomorphisms such that $\operatorname{Im}\beta$ is an abelian group and $[\operatorname{Im}\alpha, \operatorname{Im}\beta] = 1$.

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What is the relationship between $\widetilde{e}(G,N)$ and $\widetilde{e}(N,G)$ for N which is a general extensions of two groups?

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4. Study skew braces whose type is an extension of two abelian groups. Does the pattern

$$\widetilde{e}(G, N) = \widetilde{e}(N, G)$$

still hold?

Thank you for your attention!

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