Hopf orders in $(KC_p^3)^*$ over a discrete valuation ring of characteristic $p$

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1. Introduction

Let $p$ be prime, let $R$ be a discrete valuation ring of characteristic $p$ and quotient field $K$, with uniformizing parameter $\pi$ and valuation $\nu_K : K \to \mathbb{Z}$. Let $C_p^n$ denote the elementary abelian group of order $p^n$. Let $KC_p^n$ be the group ring Hopf algebra with dual Hopf algebra $(KC_p^n)^*$.

This talk concerns the structure of $R$-Hopf orders in $(KC_p^n)^*$ for $n \geq 1$. The cases $n = 1, 2$ are known; complete classifications have been given by J. Tate and F. Oort in the case $n = 1$, and G. Elder and U. in the case $n = 2$. For $n = 1$, one parameter is required to determine the Hopf order, and for $n = 2$ we require three parameters.
For arbitrary $n$, A. Koch has recently shown that Hopf orders in $(KC_p^n)^*$ are completely classified using $n(n+1)/2$ parameters.

What remains unsettled is the explicit structure of the Hopf orders in $(KC_p^n)^*$ (and their duals in $KC_p^n$).

Towards this end, we determine the algebraic structure of all Hopf orders in $(KC_p^3)^*$ and conjecture about the structure of their duals in $KC_p^3$.

We begin with a review of the $n = 1, 2$ cases.
Let $\sigma$ generate $C_p$. Then it is well-known that the group ring $KC_p$ is a $K$-Hopf algebra. Let $i \geq 0$ be an integer and let

$$\mathcal{H}_i = R \left[ \frac{\sigma - 1}{\pi^i} \right].$$

Since $(\sigma - 1)^p = 0$ in $KC_p$, it is easy to see that $\mathcal{H}_i$ is both closed under multiplication and a free $R$-module of rank $p$. Since $RC_p \subseteq \mathcal{H}_i$, we clearly have $K\mathcal{H}_i = KC_p$.

Comultiplication on $\sigma$ is grouplike, therefore, letting $x = (\sigma - 1)/\pi^i$ we have

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \pi^i x \otimes x \in \mathcal{H}_i \otimes \mathcal{H}_i.$$ 

As a result, $\mathcal{H}_i$ is a Hopf order in $KC_p$. 
Let \((KC_p)^*\) be the linear dual of \(KC_p\), and let \(\{e_i\}_{i \in \mathbb{F}_p}\) be the \(K\)-basis for \((KC_p)^*\) which is dual to the basis \(\{\sigma^j\}_{j \in \mathbb{F}_p}\) for \(KC_p\). We have \(\langle e_i, \sigma^j \rangle = \delta_{i,j}\), the Kronecker delta function.

It is well-known that \((KC_p)^*\) is a \(K\)-Hopf algebra. Multiplication in \((KC_p)^*\) is determined by \(e_i e_j = \delta_{i,j}\). Thus \(\{e_i\}_{i \in \mathbb{F}_p}\) is an orthonormal basis, and \(e_0 + e_1 + \cdots + e_{p-1}\) is the multiplicative identity. The counit is determined by \(\varepsilon(e_i) = \delta_{i,0}\), comultiplication is determined by \(\Delta(e_i) = \sum_{j \in \mathbb{F}_p} e_j \otimes e_{i-j}\), and the antipode satisfies \(S(e_i) = e_{-i}\).
Lemma 2.1. Let \( \xi_1 = \sum_{r=1}^{p-1} r e_r \in (KC_p)^* \). Then 
\( \langle \xi_1, (\sigma - 1)^j \rangle = \delta_{1,j} \) and \((RC_p)^*\) is an \( R \)-Hopf algebra with 
\((RC_p)^* = R[\xi_1]\) where \( \xi_1^p = \xi_1 \). The counit map satisfies 
\( \varepsilon(\xi_1) = 0 \), comultiplication is given as 
\( \Delta(\xi_1) = \xi_1 \otimes 1 + 1 \otimes \xi_1 \), namely \( \xi_1 \) is primitive, and the antipode satisfies 
\( S(\xi_1) = -\xi_1 \).

Proposition 2.2. Let \( i \geq 0 \) be an integer and let \( \beta = \pi^i \xi_1 \). Then 
\( R[\beta] \) is an \( R \)-Hopf algebra contained in \((RC_p)^*\) with 
\( \beta^p = \pi^{(p-1)i} \beta \); its coalgebra structure is defined by counit 
\( \varepsilon(\beta) = 0 \), comultiplication \( \Delta(\beta) = \beta \otimes 1 + 1 \otimes \beta \), and antipode 
\( S(\beta) = -\beta \). We have \( R[\beta] = \mathcal{H}_i^* \).

Theorem 2.3. [Tate-Oort] Every Hopf order in \( (KC_p)^* \) can be written as 
\( R[\beta] = R[\pi^i \xi_i] \) for some \( i \geq 0 \).

Corollary 2.4. Every Hopf order in \( KC_p \) can be written as \( \mathcal{H}_i \) for some \( i \geq 0 \).
3. Hopf orders in \((KC_p^2)^*\)

Let \(C_p^2 = \langle \sigma_1, \sigma_2 \rangle\). Then \(\{\sigma_1^a \sigma_2^b\}_{a,b \in \mathbb{F}_p}\) is a basis for \(KC_p^2\), with dual basis \(\{e_{a,b}\}_{a,b \in \mathbb{F}_p}\) for \((KC_p^2)^*\) satisfying \(\langle e_{a,b}, \sigma_1^c \sigma_2^d \rangle = \delta_{a,c} \delta_{b,d}\).

The dual \((KC_p^2)^*\) is a \(K\)-Hopf algebra. Multiplication in \((KC_p^2)^*\) is given by \(e_{a,b} e_{c,d} = \delta_{a,c} \delta_{b,d} e_{c,d}\), hence \(\{e_{a,b}\}_{a,b \in \mathbb{F}_p}\) is an orthonormal basis with \(\sum_{a,b \in \mathbb{F}_p} e_{a,b} = 1 \in (KC_p^2)^*\).

The counit map is determined by \(\varepsilon(e_{a,b}) = \delta_{a,0} \delta_{b,0}\), comultiplication is determined by \(\Delta(e_{a,b}) = \sum_{i,j \in \mathbb{F}_p} e_{i,j} \otimes e_{a-i,b-j}\), and the antipode satisfies \(S(e_{a,b}) = e_{-a,-b}\).

We identify \((KC_p^2)^*\) with \((KC_p)^* \otimes (KC_p)^*\), \(e_{a,b} \mapsto e_a \otimes e_b\).
Lemma 3.1. Let $\xi_{1,0} = \xi_1 \otimes 1$ and $\xi_{0,1} = 1 \otimes \xi_1 \in (KC_p^2)^*$. Then

$$\langle \xi_{1,0}, (\sigma_1 - 1)^j(\sigma_2 - 1)^k \rangle = \delta_{1,j}\delta_{0,k},$$

$$\langle \xi_{0,1}, (\sigma_1 - 1)^j(\sigma_2 - 1)^k \rangle = \delta_{0,j}\delta_{1,k},$$

and $(RC_p^2)^*$ is an $R$-Hopf algebra with $(RC_p^2)^* = R[\xi_{1,0}, \xi_{0,1}]$ where $\xi_{1,0}$ and $\xi_{0,1}$ satisfy $x^p = x$. On these generators, the counit satisfies $\varepsilon(x) = 0$, comultiplication is $\Delta(x) = x \otimes 1 + 1 \otimes x$, and the antipode satisfies $S(x) = -x$.

Define $\varphi(x) = x^p - x$. 
Proposition 3.2. Given integers $i_1, i_2 \geq 0$ and $\mu \in K$, let $\beta_1 = \pi^{i_1}(\xi_{1,0} - \mu \xi_{0,1})$ and $\beta_2 = \pi^{i_2}\xi_{0,1}$.

(i) If $v_K(\phi(\mu)) \geq i_2 - pi_1$, then

$$R[\beta_1, \beta_2] = R[\pi^{i_1}(\xi_{1,0} - \mu \xi_{0,1}), \pi^{i_2}\xi_{0,1}]$$

is an $R$-Hopf order in $(RC_p^2)^*$. The algebra structure of $R[\beta_1, \beta_2]$ is determined by the equations

$$\beta_1^p = \pi^{(p-1)i_1}\beta_1 - \pi^{pi_1-i_2}\phi(\mu)\beta_2,$$

and

$$\beta_2^p = \pi^{(p-1)i_2}\beta_2.$$

The coalgebra structure of $R[\beta_1, \beta_2]$ is determined on the generators, $\beta_r$, $r = 1, 2$, by counit $\varepsilon(\beta_r) = 0$, comultiplication $\Delta(\beta_r) = \beta_r \otimes 1 + 1 \otimes \beta_r$, and antipode $S(\beta_r) = -\beta_r$. In particular, the generators $\beta_1, \beta_2$ are primitive.
(ii) Let \( \beta'_1 = \pi^{i_1}(\xi_{1,0} - \mu'\xi_{0,1}) \) for some \( \mu' \in K \) satisfying 
\( v_K(\varphi(\mu')) \geq i_2 - pi_1 \). Then \( R[\beta'_1, \beta_2] \) is a Hopf algebra, and 
\( R[\beta'_1, \beta_2] = R[\beta_1, \beta_2] \) if and only if \( v_K(\mu' - \mu) \geq i_2 - i_1 \).
On the dual side, we have

**Proposition 3.3.** Let \( i_1, i_2 \geq 0, \mu \in K \), \( \sigma_1^{[\mu]} = \sum_{i=0}^{p-1} (\mu_i)(\sigma_1 - 1)^i \), and let

\[
H_{i_1, i_2, \mu} = R \left[ \frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{\sigma_2 \sigma_1^{[\mu]} - 1}{\pi^{i_2}} \right].
\]

If \( \nu_K(\varphi(\mu)) \geq i_2 - pi_1 \), then \( H_{i_1, i_2, \mu} \) is a Hopf order in \( KC_p^2 \).

**Theorem 3.4.** Let \( H_{i_1, i_2, \mu} \) be as in Proposition 3.3, then

\[
H_{i_1, i_2, \mu}^* = R[\beta_1, \beta_2] = R[\pi^{i_1}(\xi_{1,0} - \mu \xi_{0,1}), \pi^{i_2} \xi_{0,1}].
\]
We now show that every Hopf order in \((KC_p^2)^*\) is of the form

\[ R[\beta_1, \beta_2] = R[\pi^{i_1}(\xi_{1,0} - \mu \xi_{0,1}), \pi^{i_2} \xi_{0,1}] . \]

Recall \(C_p^2 = \langle \sigma_1, \sigma_2 \rangle\), and let \(\mathcal{H}\) be an \(R\)-Hopf order in \(KC_p^2\). Let \(C_p^2 \rightarrow C_p^2/\langle \sigma_1 \rangle\) denote the canonical surjection with \(C_p^2/\langle \sigma_1 \rangle \cong \langle \bar{\sigma}_2 \rangle\) where \(\bar{\sigma}_2 = \sigma_2 \langle \sigma_1 \rangle\). There exists a short exact sequence

\[
R \rightarrow \mathcal{H}_{i_1} \rightarrow \mathcal{H} \rightarrow \mathcal{H}_{i_2} \rightarrow R, \tag{1}
\]

where \(\mathcal{H}_{i_1} = R[(\sigma_1 - 1)/\pi^{i_1}]\) and \(\mathcal{H}_{i_2} = R[(\bar{\sigma}_2 - 1)/\pi^{i_2}]\), for some \(i_1, i_2 \geq 0\).

We dualize (1) to obtain the short exact sequence

\[
R \rightarrow \mathcal{H}_{i_2}^* \rightarrow \mathcal{H}^* \rightarrow \mathcal{H}_{i_1}^* \rightarrow R. \tag{2}
\]
We next translate into the language of group schemes. Let

\[ D_{i_1}^* = \text{Spec } \mathcal{H}_{i_1}^*, \quad D^* = \text{Spec } \mathcal{H}^*, \text{ and } D_{i_2}^* = \text{Spec } \mathcal{H}_{i_2}^*. \]

Classifying all Hopf orders \( \mathcal{H} \) in (1), or \( \mathcal{H}^* \) in (2), is the same as classifying all finite group schemes \( D^* \) that fit into the short exact sequence of group schemes

\[ 0 \rightarrow D_{i_1}^* \rightarrow D^* \rightarrow D_{i_2}^* \rightarrow 0, \tag{3} \]

and which are represented by an \( R \)-Hopf order in \( (KC_p^2)^* \). In other words, we compute the subgroup \( \text{Ext}^1_{gt}(D_{i_2}^*, D_{i_1}^*) \) of generically trivial extensions within the full extension group \( \text{Ext}^1(D_{i_2}^*, D_{i_1}^*) \).
To this end, observe that the polynomial ring $R[x]$ with counit $\varepsilon(x) = 0$, comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$ and antipode $S(x) = -x$ represents the additive group scheme $\mathbb{G}_a$.

For $i_1 \geq 0$, the $R$-algebra map $\psi : R[x] \to R[x]$ determined by $\psi(x) = x^p - \pi^{(p-1)i_1}x$ is a homomorphism of Hopf algebras, and so, there exists a homomorphism of $R$-group schemes

$$\Psi : \mathbb{G}_a \to \mathbb{G}_a,$$

defined by $\Psi(g)(x) = g(\psi(x))$ for $g \in \mathbb{G}_a$. The kernel of $\Psi$ is represented by the $R$-Hopf order $R[x]/(\psi(x)) \cong \mathcal{H}_{i_1}^*$ in $(KC_p)^*$, thus there is a short exact sequence of group schemes

$$0 \to \mathbb{D}_{i_1}^* \xrightarrow{i} \mathbb{G}_a \xrightarrow{\psi} \mathbb{G}_a \to 0. \quad (4)$$
From (4), we obtain the long exact sequence:

$$
\text{Hom}(D_{i_2}^*, G_a) \xrightarrow{\Psi} \text{Hom}(D_{i_2}^*, G_a) \xrightarrow{\omega} \text{Ext}^1(D_{i_2}^*, D_{i_1}^*) \xrightarrow{\iota} \text{Ext}^1(D_{i_2}^*, G_a),
$$

with connecting homomorphism $\omega$, which induces the map $\rho$ in the exact sequence

$$
0 \rightarrow \text{coker}(\Psi : \text{Hom}(D_{i_2}^*, G_a) \xrightarrow{\iota}) \xrightarrow{\rho} \text{Ext}^1(D_{i_2}^*, D_{i_1}^*) \xrightarrow{\iota} \text{Ext}^1(D_{i_2}^*, G_a).
$$

Tensoring with $K$ and considering kernels, we obtain the exact sequence

$$
0 \rightarrow \text{coker}(\Psi : \text{Hom}(D_{i_2}^*, G_a) \xrightarrow{\iota})_{gt} \xrightarrow{\rho} \text{Ext}^1_{gt}(D_{i_2}^*, D_{i_1}^*) \xrightarrow{\iota} \text{Ext}^1_{gt}(D_{i_2}^*, G_a).
$$

(5)
Proposition 3.5. There is an isomorphism

\[ \rho : \text{coker}(\psi : \text{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a))^\bigcirc )_{gt} \to \text{Ext}^1_{gt}(\mathbb{D}^*_{i_2}, \mathbb{D}^*_{i_1}). \]

**Proof.** Our plan is to show that \( \text{Ext}^1_{gt}(\mathbb{D}^*_{i_2}, \mathbb{G}_a) = 0 \) in (5). To this end, we use a first quadrant spectral sequence to show that \( \text{Ext}^1(\mathbb{D}^*_{i_2}, \mathbb{G}_a) \cong H^2(\mathbb{D}^*_{i_2}, \mathbb{G}_a). \) With this characterization, we then form the complex of morphisms

\[ \text{Mor}_0((\mathbb{D}^*_{i_2})^{r-1}, X) \xrightarrow{\partial_{r-1}} \text{Mor}_0((\mathbb{D}^*_{i_2})^r, X) \xrightarrow{\partial_r} \text{Mor}_0((\mathbb{D}^*_{i_2})^{r+1}, X) \xrightarrow{\partial_{r+1}}, \]

and compute directly that

\[ H^2_0(\mathbb{D}^*_{i_2}, \mathbb{G}_a) \to H^2_0(K \otimes_R \mathbb{D}^*_{i_2}, K \otimes_R \mathbb{G}_a) \]

is an injection, thus \( H^2_0(\mathbb{D}^*_{i_2}, \mathbb{G}_a)_{gt} \cong \text{Ext}^1_{gt}(\mathbb{D}^*_{i_2}, \mathbb{G}_a) = 0 \) is trivial. \( \square \)
In order to compute the elements of $\text{Ext}^1_{gt}(\mathbb{D}^*_{i_2}, \mathbb{D}^*_{i_1})$, explicitly, we need to characterize $\text{coker}(\Psi_1 : \text{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a)^{\circ})_{gt}$.

**Proposition 3.6.** The $\text{coker}(\Psi_1 : \text{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a)^{\circ})_{gt}$ is isomorphic to the additive subgroup of $K/(\mathbb{F}_p + P^{i_2-i_1})$ represented by those elements $\mu \in K$ satisfying $\wp(\mu) \in P^{i_2-pi_1}$.

**Proof.** Each element of $\text{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a)$ corresponds to a $R$-Hopf algebra homomorphism $R[x] \rightarrow \mathcal{H}^*_{i_2}$, and since $x$ is primitive, elements of $\text{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a)$ correspond to $\text{Prim}(\mathcal{H}^*_{i_2})$, the primitive elements in $\mathcal{H}^*_{i_2}$. We have $\mathcal{P} = \text{Prim}(\mathcal{H}^*_{i_2}) = R\beta_2$ where $\beta_2 = \pi^{i_2} \xi_{0,1}$.

The generically trivial elements in the cokernel $\text{coker}(\Psi_1 : \text{Hom}(\mathbb{D}^*_{i_2}, \mathbb{G}_a)^{\circ})$ correspond to elements of 

$$(\psi(K \otimes_R \mathcal{P}) \cap \mathcal{P})/\psi(\mathcal{P}).$$
Elements of $K \otimes_R \mathcal{P}$ can be expressed as $\mu \pi^{i_1} \xi_{0,1}$ for some $\mu \in K$, and an element of $\psi(K \otimes_R \mathcal{P})$ can be written

$$\phi(\mu) \pi^{p i_1} \xi_{0,1} = \psi(\mu \pi^{i_1} \xi_{0,1}).$$

An element of $\psi(K \otimes_R \mathcal{P})$ lies in $\mathcal{P}$ precisely when $\phi(\mu) \in P^{i_2-p i_1}$. It is zero in the quotient $(\psi(K \otimes_R \mathcal{P}) \cap \mathcal{P})/\psi(\mathcal{P})$ precisely when $\mu \in \mathbb{F}_p + P^{i_2-i_1}$. \hfill \Box
Theorem 3.7. Each class \([E]\) in \(\text{Ext}^1_{gt}(D^*_i, D^*_j)\) corresponds to a short exact sequence

\[
E_\mu : 0 \to D^*_i \to \text{Spec } R[\pi^{i_1}(\xi_{1,0} - \mu \xi_{0,1}), \pi^{i_2} \xi_{0,1}] \to D^*_j \to 0
\]

where \(\mu \in K\) represents a coset in \(K/(\mathbb{F}_p + P^{i_2-i_1})\) that satisfies \(\nu_K(\varphi(\mu)) \geq i_2 - pi_1\).

Proof. Let \([E] \in \text{Ext}^1_{gt}(D^*_i, D^*_j),\)

\[
E : 0 \to D^*_i \to D^* \to D^*_j \to 0.
\]

By Proposition 3.5, \(\rho^{-1}([E]) = [h]\) is a class in the cokernel represented by a homomorphism \(h : D^*_j \to \mathbb{G}_a\) and is determined by a Hopf algebra map \(x \mapsto \varphi(\mu)\pi^{pi_1} \xi_{0,1} = \varphi(\mu)\pi^{pi_1} \xi_{0,1}\) for some \(\mu \in K\) with \(\nu_K(\varphi(\mu)) \geq i_2 - pi_1\).
We compute the representing Hopf algebra $\mathcal{H}_h^*$ of $\mathbb{D}_h^* = \mathbb{D}^*$.

Translating to Hopf algebras, we have the push-out diagram

$$
\begin{array}{ccc}
\mathcal{H}_h^* & \leftarrow & R[x] \\
\uparrow & & \psi \uparrow \\
\mathcal{H}_{i_2}^* & \leftarrow & R[x],
\end{array}
$$

with $\alpha(x) = \varphi(\mu) \pi^{i_1} \xi_{0,1} = \psi(\mu \pi^{i_1} \xi_{0,1})$. Thus,

\[
\mathcal{H}_h^* = \frac{(R[\pi^{i_2} \xi_{0,1}] \otimes_R R[x])}{(\varphi(\mu) \pi^{i_1} \xi_{0,1} \otimes 1 + 1 \otimes \psi(x))} \\
\cong \frac{R[i^{i_2} \xi_{0,1}] [x]}{(\psi(x) + \varphi(\mu) \pi^{i_1} \xi_{0,1})} \\
= \frac{R[i^{i_2} \xi_{0,1}] [x]}{(\psi(x) + \psi(\mu \pi^{i_1} \xi_{0,1}))} \\
= \frac{R[i^{i_2} \xi_{0,1}] [x]}{(\psi(x + \mu \pi^{i_1} \xi_{0,1}))}.
\]

With $x \mapsto \pi^{i_1} \xi_{1,0}$, under $R[x] \to R[x]/\psi(x) \cong R[\pi^{i_1} \xi_{1,0}]$, one obtains

\[
\mathcal{H}_h^* \cong R[\pi_1 (\xi_{1,0} - \mu \xi_{0,1}), \pi^{i_2} \xi_{0,1}].
\]
And as we have seen,

\[ \mathcal{H}_h \cong R[\pi_i^1(\xi_{1,0} - \mu \xi_{0,1}), \pi_i^2 \xi_{0,1}]^* \]

\[ \cong \mathcal{H}_{i_1, i_2, \mu} \]

\[ = R \left[ \frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{\sigma_2 \sigma_1^{[\mu]} - 1}{\pi^{i_2}} \right]. \]

Thus every \( R \)-Hopf order in \( KC_p^2 \) is of the form \( \mathcal{H}_{i_1, i_2, \mu} \).
4. Hopf orders in \((KC_p^3)^*\)

How much of the method of the \(n = 2\) case carries over to \(n \geq 3\)?

Let \(C_p^3 = \langle \sigma_1, \sigma_2, \sigma_3 \rangle\), \(\bar{\sigma}_2 = \sigma_2\langle \sigma_1 \rangle\), \(\bar{\sigma}_3 = \sigma_3\langle \sigma_1 \rangle\), and let

\[
R \to R \left[ \frac{\sigma_1 - 1}{\pi^{i_1}} \right] \to \mathcal{H} \to R \left[ \frac{\bar{\sigma}_2 - 1}{\pi^{i_2}}, \frac{\bar{\sigma}_3 \bar{\sigma}_2[\mu] - 1}{\pi^{i_3}} \right] \to R
\]

be a short exact sequence of \(R\)-Hopf orders, \(\mathcal{H} \subseteq KC_p^3\), dualizing as

\[
R \to R[\pi^{i_2}(\xi_{0,1,0} - \mu \xi_{0,0,1}), \pi^{i_3} \xi_{0,0,1}] \to \mathcal{H}^* \to R[\pi^{i_1} \xi_{1,0,0}] \to R,
\]

where \(\xi_{i,j,k} = \xi_i \otimes \xi_j \otimes \xi_k\).
Applying $\text{Spec}$ gives

$$0 \to \mathcal{D}_{i_1}^* \to \mathcal{D}^* \to \mathcal{D}_{i_2, i_3, \mu}^* \to 0,$$

where

$$\mathcal{D}_{i_2, i_3, \mu}^* = \text{Spec } R[\pi^{i_2}(\xi_{0,1,0} - \mu \xi_{0,0,1}), \pi^{i_3} \xi_{0,0,1}].$$

Note: $\mathcal{D}_{i_2, i_3, \mu}^*$ plays the role of $\mathcal{D}_{i_2}^*$ in the $n = 2$ case.

We want to classify short exact sequences of the form (6). Most of the results in the $n = 2$ case extend easily, in fact:

**Proposition 4.1.** There is an isomorphism

$$\rho : \text{coker}(\Psi : \text{Hom}(\mathcal{D}_{i_2, i_3, \mu}^*, \mathbb{G}_a) \otimes )_{gt} \to \text{Ext}^1_{gt}(\mathcal{D}_{i_2, i_3, \mu}^*, \mathcal{D}_{i_1}^*).$$
So it is a matter of computing $coker(\psi : \text{Hom}(D^*_{i_2,i_3,\mu,G_a}) \rightarrow \mathcal{O})_{gt}$.

To this end, we see that elements of $\text{Hom}(D^*_{i_2,i_3,\mu,G_a})$ correspond to Hopf maps $R[x] \rightarrow R[\pi^{i_2}(\xi_{0,1,0} - \mu \xi_{0,0,1}), \pi^{i_3} \xi_{0,0,1}]$ given as $x \mapsto a$, where $a \in \mathcal{P} = \text{Prim}(R[\pi^{i_2}(\xi_{0,1,0} - \mu \xi_{0,0,1}), \pi^{i_3} \xi_{0,0,1}])$.

Ultimately, we need to compute

$$(\psi(K \otimes \mathcal{P}) \cap \mathcal{P})/\psi(\mathcal{P}).$$

Now, $K \otimes \mathcal{P} = K \xi_{0,1,0} + K \xi_{0,0,1}$, and elements of $K \otimes \mathcal{P}$ can be written

$$\omega \pi^{i_1} \xi_{0,1,0} + \theta \pi^{i_1} \xi_{0,0,1}$$

for $\omega, \theta \in K$. 
Thus an element in $\psi(K \otimes \mathcal{P})$ is

$$\psi(\omega \pi^{i_1} \xi_{0,1,0} + \theta \pi^{i_1} \xi_{0,0,1}) = \wp(\omega)\pi^{p i_1} \xi_{0,1,0} + \wp(\theta)\pi^{p i_1} \xi_{0,0,1}.$$  

This element is in $\mathcal{P}$ under certain conditions on $\wp(\omega)$ and $\wp(\theta)$; it is in $\psi(\mathcal{P})$ under certain conditions on $\omega$ and $\theta$.

We determine these conditions.

Note that $\wp(\omega)\pi^{p i_1} \xi_{0,1,0} + \wp(\theta)\pi^{p i_1} \xi_{0,0,1} \in \mathcal{P}$ if and only if

$$\langle \wp(\omega)\pi^{p i_1} \xi_{0,1,0} + \wp(\theta)\pi^{p i_1} \xi_{0,0,1}, \mathcal{H}_{i_2,i_3,\mu} \rangle \subseteq R.$$
Since

\[ \bar{\sigma}_3 \bar{\sigma}_2^{[\mu]} - 1 = (\bar{\sigma}_3 - 1 + 1)\bar{\sigma}_2^{[\mu]} - 1 \]

\[ = (\bar{\sigma}_3 - 1) \sum_{i=0}^{p-1} \left( \binom{\mu}{i} (\bar{\sigma}_2 - 1)^i \right) + \sum_{i=1}^{p-1} \left( \binom{\mu}{i} (\bar{\sigma}_2 - 1)^i \right) \]

\[ = (\bar{\sigma}_3 - 1) \left( 1 + \sum_{i=1}^{p-1} \left( \binom{\mu}{i} (\bar{\sigma}_2 - 1)^i \right) \right) \]

\[ + \mu(\bar{\sigma}_2 - 1) + \sum_{i=2}^{p-1} \left( \binom{\mu}{i} (\bar{\sigma}_2 - 1)^i \right) \]

\[ = (\bar{\sigma}_3 - 1) + \mu(\bar{\sigma}_2 - 1) + \sum_{i=2}^{p-1} \left( \binom{\mu}{i} (\bar{\sigma}_2 - 1)^i \right) \]

\[ + \sum_{i=1}^{p-1} \left( \binom{\mu}{i} (\bar{\sigma}_3 - 1)(\bar{\sigma}_2 - 1)^i \right), \]
It suffices to show that

\[ \langle \varphi(\omega) \pi^{p_1} \xi_{0,1,0} + \varphi(\theta) \pi^{p_1} \xi_{0,0,1}, \bar{\sigma}_2 - 1 \rangle \in \pi^{i_2} R, \]

and

\[ \langle \varphi(\omega) \pi^{p_1} \xi_{0,1,0} + \varphi(\theta) \pi^{p_1} \xi_{0,0,1}, (\bar{\sigma}_3 - 1) + \mu(\bar{\sigma}_2 - 1) \rangle \in \pi^{i_3} R, \]

The first condition is

\[ \nu_K(\varphi(\omega)) \geq i_2 - p_i, \]

and the second condition is

\[ \nu(\varphi(\theta) + \mu \varphi(\omega)) \geq i_3 - p_i. \]

Note: if \( \nu_K(\mu) \leq 0 \), then \( \nu_K(\mu) \geq \frac{i_3}{p} - i_2 \). Thus,

\[ \nu_K(\mu \varphi(\omega)) \geq \frac{i_3}{p} - i_2 + i_2 - p_i = \frac{i_3}{p} - p_i, \]

and so,

\[ \nu(\varphi(\theta)) \geq \frac{i_3}{p} - p_i. \]
Here is the classification result.

**Theorem 4.2.** Each class $[E]$ in $\text{Ext}^1_{gt}(\mathbb{D}^*_{i_2,i_3,\mu}, \mathbb{D}^*_{i_1})$ corresponds to a short exact sequence

$$E_{\omega,\theta} : 0 \to \mathbb{D}^*_{i_1}$$

$\longrightarrow \text{Spec } R[\pi^{i_1}(\xi_{1,0,0} - \omega \xi_{0,1,0} - \theta \xi_{0,0,1}), \pi^{i_2}(\xi_{0,1,0} - \mu \xi_{0,0,1}), \pi^{i_3} \xi_{0,0,1}]$

$\longrightarrow \mathbb{D}^*_{i_2,i_3,\mu} \to 0$

where $\mu, \omega, \theta \in K$ satisfy

$$\nu_K(\varphi(\mu)) \geq i_3 - pi_2, \quad \nu(\varphi(\omega)) \geq i_2 - pi_1, \quad \nu_K(\varphi(\theta)) \geq \frac{i_3}{p} - pi_1.$$
Finally, we have a conjecture.

**Conjecture 4.3. The Hopf order**

\[
R[\pi^i_1(\xi_{1,0,0} - \omega \xi_{0,1,0} - \theta \xi_{0,0,1}), \pi^i_2(\xi_{0,1,0} - \mu \xi_{0,0,1}), \pi^i_3 \xi_{0,0,1}]
\]

in \((KC^3_p)^*\) is the linear dual of the Hopf order

\[
R \left[ \frac{\sigma_1 - 1}{\pi^i_1}, \frac{\sigma_2 \sigma_1^{[\omega]} - 1}{\pi^i_2}, \frac{\sigma_3 \sigma_1^{[\theta]}(\sigma_2 \sigma_1^{[\omega]})^{[\mu]} - 1}{\pi^i_3} \right]
\]

in \(KC^3_p\).

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