Hopf-Galois Structures on Quaternionic Extensions

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Why study (tame) quaternionic extensions?

- Quaternionic extensions of number fields have been important in the history of Galois module structure.
- There exist tamely ramified quaternionic extensions $L/\mathbb{Q}$ which have local, but not global, normal integral bases.
- We might look to these for examples of cases where a non-classical Hopf-Galois structure provides a better description of the algebraic integers.
- Tameness is not used in the work displayed here, but will be useful for eventual deductions on freeness.

Our extension

- Let $L/\mathbb{Q}$ be a Galois extension with Galois group, G, equal to the quaternion group of order 8.
- Then there exists a unique biquadratic subextension, $K/\mathbb{Q}$, that is Galois with group $\bar{G} \cong C_2 \times C_2$.
- Let $G = \langle \sigma, \tau \mid \sigma^4 = 1, \sigma^2 = \tau^2, \sigma\tau = \tau\sigma^{-1} \rangle$.
- Let $K = \mathbb{Q}(\alpha, \beta)$ where $\bar{\sigma}(\alpha) = -\bar{\tau}(\alpha) = \alpha$ and $\bar{\sigma}(\beta) = -\bar{\tau}(\beta) = -\beta$

Greither & Pareigis

Theorem (Greither & Pareigis, 1987)

There is a bijection between regular subgroups N of Perm(G) *normalised by $\lambda(G)$ and Hopf-Galois structures on $L/\mathbb{Q}$ defined by $N \leftrightarrow L[N]^G$.*

- G acts on N via conjugation by $\lambda(G)$.
- A subgroup N of Perm(G) is said to be regular if $|N| = |G|$ and $\mathrm{stab}_N(g)$ is trivial for all $g \in G$.
- Perm(G) is large so getting at possible N is difficult.

Byott's translation theorem

Theorem (Byott, 1996)

Let N be a group of order $|G|$. There is a bijection between

$$\mathcal{N} = \{\alpha : N \to \mathrm{Perm}(G) \text{ a 1-1 homomorphism s.t. } \alpha(N) \text{ is regular}\}$$

and

$$\mathcal{G} = \{\beta : G \to \mathrm{Perm}(N) \text{ a 1-1 homomorphism s.t. } \beta(G) \text{ is regular}\}.$$

Under this bijection if $\alpha, \alpha' \in \mathcal{N}$ correspond to $\beta, \beta' \in \mathcal{G}$ respectively, then:

- *$\alpha(N) = \alpha'(N)$ iff $\beta(G)$ is conjugate to $\beta'(G)$ by an element of $\mathrm{Aut}(N)$,*
- *$\alpha(N)$ is normalised by $\lambda(G) \subset \mathrm{Perm}(G)$ iff $\beta(G) \subseteq \mathrm{Hol}(N) \cong N \rtimes \mathrm{Aut}(N)$.*

Elementary abelian example

- We can view C_2^3 as $\mathbb{F}_2^3$ so that $\text{Aut}(N) \cong GL_3(\mathbb{F}_2)$.
- We find, using Sylow subgroup theory, 14 regular subgroups of $\text{Hol}(N)$ isomorphic to $G \cong Q_8$, represented by M.
- We then have $|\text{Aut}(Q_8)|/|\text{Aut}(N)| \cdot 14 = 14/7 = 2$ regular subgroups of $\text{Perm}(G)$.
- This comes from two collections, each of 7 of the subgroups, that are conjugate to each other.
- We construct $\alpha = C((\beta(G) \cdot 1_N)^{-1}) \cdot \lambda_N$ where $\beta : G \to M$ once for each collection representative.
- Write down the $N_k = \alpha_{M_k}(N)$ and apply Greither-Pareigis for the Hopf-Galois structures $L[N_k]^G$.

Final count

Abelian types

$C_2 \times C_2 \times C_2$ type

- 14 $M \subset \mathrm{Hol}(N)$,
- 2 $N_k \subset \mathrm{Perm}(G)$.

C_8 type

- 1 $M \subset \mathrm{Hol}(N)$,
- 6 $N_k \subset \mathrm{Perm}(G)$.

$C_4 \times C_2$ type

- 2 $M \subset \mathrm{Hol}(N)$,
- 6 $N_k \subset \mathrm{Perm}(G)$.

Non-abelian types

Q_8 type

- 2 $M \subset \mathrm{Hol}(N)$,
- 2 $N_k \subset \mathrm{Perm}(G)$ (classical and standard non-classical)

D_4 type

- 2 $M \subset \mathrm{Hol}(N)$,
- 6 $N_k \subset \mathrm{Perm}(G)$ (3 from each).

Boltje & Bley

Theorem (Boltje & Bley, 1999)

For N abelian

- *let* $\chi_1, \ldots, \chi_s \in \widehat{N}$ *be a set of representatives of the G-orbits for* $\widehat{N}$.
- *Let* $\widehat{L_k}$ *denote the fixed field of* $S_k = \mathrm{stab}(\chi_k)$.

Then

- $L[N]^G \cong \prod_{k=1}^{s} \widehat{L_k}$,
- *the maximal order of* $L[N]^G$ *is* $\mathcal{M} \cong \prod_{k=1}^{s} \mathcal{O}_{\widehat{L_k}}$.

Elementary abelian example

- Find G-orbits of $\widehat{N}$, choose representatives, and write down stabilisers in each case.
- In both of these cases the orbit structure is the same, as are the stabilisers: 4 trivial orbits with stabiliser G and one of size 4 with stabiliser $\langle\sigma^2\rangle$.
- So we find

$$L[N_i]^G \cong \mathbb{Q}^4 \times K.$$

Quaternionic example

Class	$\{1\}$	$\{\sigma^2\}$	$\{\sigma, \sigma^3\}$	$\{\tau, \sigma^2\tau\}$	$\{\sigma\tau, \sigma^3\tau\}$
χ_0	1	1	1	1	1
χ_1	1	1	1	-1	-1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	2	-2	0	0	0

We have that;

- the four 1-dimensional characters are realisable over $\mathbb{R}$ so have values in $\mathbb{Q}$;
- no orbits mix elements of different conjugacy classes.

Thus the G action on the idempotents of the four 1-dimensional characters is trivial and they are, of course, orthogonal to the other idempotents. So we get a copy of $\mathbb{Q}$ for each of these.

Quaternionic example continued

- χ_4 is realisable over $\mathbb{Q}(i)$ so let $F = L(i)$ so that $F[N] \cong F^4 \times \mathrm{Mat}_{2\times 2}(F)$.
- Let Γ be the Galois group of F/L. Then $F[N]^\Gamma = L[N]$.

Notice the 4 F slices will fix to the $\mathbb{Q}$ slices from before.
For the dimension 4 slice

- let $e = \frac{1}{4}(1 - \sigma^2)$, the idempotent corresponding to χ_4.
- Then $\{e, e\sigma, e\tau, e\sigma\tau\}$ is a $\mathbb{Q}$-basis for the dimension 4 slice.
- The multiplication table for these is that of the quaternions $\mathbb{H}$.

The action of G is trivial for $N = \rho(G)$. For $N = \lambda(G)$ the action gives us that an element of the dimension 4 slice is fixed by G if and only if it has the form

$$a_0 e + a_1 \alpha e\sigma + a_2 \beta e\tau + a_3 \alpha\beta e\sigma\tau.$$

Thus

$$L[\rho(G)]^G \cong \mathbb{Q}^4 \times \mathbb{H} \quad \& \quad L[\lambda(G)]^G \cong \mathbb{Q}^4 \times \mathbb{Q}(\alpha i, \beta j).$$

Descriptions

For $\mu \in \{\alpha, \beta, \alpha\beta\}$ determined by the stabiliser in each case we have

$C_2 \times C_2 \times C_2$ type

$L[N]^G \cong \mathbb{Q}^4 \times K$

$C_4 \times C_2$ type

$L[N]^G \cong \mathbb{Q}^4 \times \mathbb{Q}(\mu i)^2$

C_8 type

$L[N]^G \cong \mathbb{Q}^2 \times \mathbb{Q}(\mu i) \times \mathbb{Q}(\sqrt{2}, \mu i)$

Q_8 type

$L[\rho(G)]^G \cong \mathbb{Q}^4 \times \mathbb{H}$
$L[\lambda(G)]^G \cong \mathbb{Q}^4 \times \mathbb{Q}(\alpha i, \beta j)$

D_4 type

$L[N]^G \cong \mathbb{Q}^4 \times \mathbb{Q}(\alpha i, \beta j)$ in three cases
$L[N]^G \cong \mathbb{Q}^4 \times \mathbb{Q}(i, \mu j)$ in the other cases

Observations

- All of the cases in each type come from the same subgroup of the respective automorph.
- The D_4 type has two different algebra descriptions corresponding to how they arose from the holomorph.
- Three of the Hopf-Galois structures of type D_4 are isomorphic as algebras to the structure $L[\lambda(G)]^G$.
- The classical Hopf-Galois structure and the standard non-classical structure are not isomorphic as algebras.

Thank you

Thank you for listening!