Short Exact Sequences of Hopf-Galois Extensions

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Hopf-Galois Theory

An extension K/k is Hopf-Galois if there is a k-Hopf algebra H and a k-algebra homomorphism $\mu : H \to End_k(K)$ such that

- $\mu(ab) = \sum_{(h)} \mu(h_{(1)}(a)\mu(h_{(2)})(b)$
- $K^H = \{a \in K \mid \mu(h)(a) = \epsilon(h)a \ \forall h \in H\} = k$
- μ induces $I \otimes \mu : K\#H \overset{\cong}{\to} End_k(K)$

If K/k is Galois with $G = Gal(K/k)$ then, by linear independence of characters, the elements of G are a k-basis for $End_k(K)$ whence there exists a natural map:

$$H = k[G] \stackrel{\mu}{\to} End_k(K)$$

which induces

$$I \otimes \mu : K\#H \stackrel{\cong}{\to} End_k(K)$$

For the group ring $k[G]$ the endomorphisms arise as linear combinations of the automorphisms given by the elements of G.

Hopf-Galois theory is a generalization of ordinary Galois theory in several ways.

- One can put Hopf Galois structure(s) on separable field extensions K/k which aren't classically Galois. e.g. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$
- Moreover, one can take an extension K/k which *is* Galois with group G (hence Hopf-Galois for $H = k[G]$) and also find *other* Hopf algebras which act besides $k[G]$.

In this talk we will be focusing on the case where K/k is already a Galois extension.

Both cases are covered by the Greither-Pareigis enumeration and the formulation for the latter is as follows:

- K/k finite Galois extension with $G = Gal(K/k)$.

G acting on itself by left translation yields an embedding

$$\lambda : G \hookrightarrow B = Perm(G)$$

Definition: $N \leq B$ is *regular* if N acts transitively and fixed point freely on G.

Theorem

[3] The following are equivalent:

- *There is a k-Hopf algebra H such that K/k is H-Galois*
- *There is a regular subgroup $N \leq B$ s.t. $\lambda(G) \leq Norm_B(N)$ where N yields $H = (K[N])^G$.*

Short Exact Sequences

In ordinary Galois theory, if K/k is Galois with $G = Gal(K/k)$ and $G' \triangleleft G$ with $K^{G'}$ the corresponding intermediate field, then $K^{G'}/k$ is Galois with group G/G'. Also, of course, one has the exact sequence of groups

$$1 \to G' \to G \to G/G' \to 1$$

One wonders what an analogous formulation would look like for Hopf-Galois structures.

Since $G' \triangleleft G$ then by [2, Prop 4.14]

$$k \to k[G'] \to k[G] \to k[G/G'] \to k$$

is a short exact sequence of k-Hopf algebras.

But in terms of the actions on the relevant (intermediate) fields, these are not exactly the Hopf algebras that act to make the given field extensions Hopf-Galois.

If we look at the Hopf-Galois actions induced by the Galois groups G', G, and G/G' then the Hopf algebras are group rings

$$K/K^{G'} \text{ is acted on by } (K[\rho(G')])^{\lambda(G')} \cong K^{G'}[G']$$
$$\text{where } \rho(G'), \lambda(G') \leq Perm(G')$$

$$K/k \text{ is acted on by } (K[\rho(G)])^{\lambda(G)} \cong k[G]$$

$$K^{G'}/k \text{ is acted on by } (K^{G'}[\rho(G/G')])^{\lambda(G/G')} \cong k[G/G']$$
$$\text{where } \lambda(G/G'), \rho(G/G') \leq Perm(G/G')$$

The latter two Hopf algebras are defined over k but the first is not, which is not unexpected since it is acting with respect to the ground field $K^{G'}$.

Since $\rho(G')$ is plainly normalized by $\rho(G)$ and $\lambda(G)$ then $(K[\rho(G')])^{\lambda(G)}$ is an 'admissible sub-algebra' of $(K[\rho(G)])^{\lambda(G)}$ where by [1, Theorem 7.6] (Chase and Sweedler)

$$(K[\rho(G')])^{\lambda(G')} \cong K^{G'} \otimes (K[\rho(G')])^{\lambda(G)}$$

Part of the simplicity of the 'classical' case above is that the Hopf-Algebras are group rings, which is due to the fact that $\lambda(G)$ centralizes $\rho(G)$ and concordantly $\rho(G')$ so that the descent data is only acting on the scalars.

For K/k Hopf-Galois under the action of $H_N = (K[N])^{\lambda(G)}$, where N is a regular subgroup of $Perm(G)$ normalized by $\lambda(G)$, we would like to consider $P \triangleleft N$, also normalized by $\lambda(G)$ and the Hopf-Galois structures (if any) arising from P and N/P.

Since P is normalized by N, then by Chase and Sweedler [1, Theorem 7.6], $H_P = (K[P])^{\lambda(G)}$ is an *admissible* k-sub Hopf algebra of $H_N = (K[N])^{\lambda(G)}$ which fixes a subfield F, and that $F \otimes H_P$ acts to make K/F Hopf-Galois.

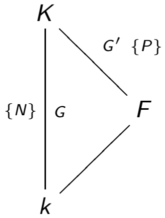
Furthermore, since F is an intermediate field between K and k then $F = K^{G'}$ for some $G' \leq G = Gal(K/k)$.

[**Side Question:** Is it possible to deduce G' in terms of P directly, in a field independent way? After all, G' is embedded as a subgroup of $\rho(G)$ (the classical action!) in $Perm(G)$ and $P \leq N \leq Perm(G)$ where, for example, one *must* have $|G'| = |P|$. Perhaps this might de-mystify the FTGT correspondence between subfields and sub Hopf-algebras.]

Since K/F is Galois with group G' and acted on by $F \otimes H_P$ then how does it fit within the Greither-Pareigis framework?

As observed by Crespo in [5, Prop. 8], $F \otimes H_P = (K[P])^{\lambda(G')}$.

That is, P is embedded as a regular subgroup of $Perm(G')$ and normalized by $\lambda(G')$.



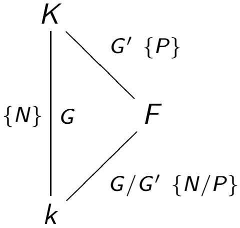
However, since $N \leq Perm(G)$ and $P \leq Perm(G') \leq Perm(G)$ then P is also embedded in $Perm(G)$ as a semi-regular (fixed point free) subgroup. (Recall that a proper subgroup of a regular permutation group is semi-regular.)

This parallels the relationship between $\lambda(G') \leq Perm(G')$ as a regular subgroup and $\lambda(G') \leq Perm(G)$ as a semi-regular subgroup.

Indeed, there is useful (I dare say critical) information to be obtained by examining the regularity and semi-regularity of these subgroups.

For the 'quotient' piece, namely the construction of a Hopf-Galois structure on F/k corresponding to N/P, some care must be taken since, applying Greither-Pareigis naively we might want N/P to be normalized by $\lambda(G/G') \leq Perm(G/G')$.

However, this pre-supposes that $G' \leq G$ is actually a normal, but this need not be the case, even though $P \triangleleft N$.



As an example, suppose G and N below correspond to $\rho(G)$ and N embedded in $Perm(G)$, although here we have S_{24} instead of $Perm(G)$, which actually doesn't matter.

$$G = \langle (1,2)(3,13)(4,8)(5,7)(6,9)(10,21)(11,20)(12,16)(14,18)(15,17)(19,24)(22,23),$$
$$(1,3,9)(2,6,13)(4,11,23)(5,19,17)(7,15,24)(8,22,20)(10,18,12)(14,21,16),$$
$$(1,4)(2,7)(3,10)(5,12)(6,14)(8,16)(9,17)(11,19)(13,20)(15,22)(18,23)(21,24)$$
$$(1,5)(2,8)(3,11)(4,12)(6,15)(7,16)(9,18)(10,19)(13,21)(14,22)(17,23)(20,24)\rangle$$
$$\cong S_4 \text{ regular}$$

$$N = \langle (1,6,9,2,3,13)(4,15,23,7,11,24)(5,22,17,8,19,20)(10,21,12,14,18,16),$$
$$(1,9,3)(2,13,6)(4,23,11)(5,17,19)(7,24,15)(8,20,22)(10,12,18)(14,16,21),$$
$$(1,4)(2,7)(3,10)(5,12)(6,14)(8,16)(9,17)(11,19)(13,20)(15,22)(18,23)(21,24),$$
$$(1,12)(2,16)(3,19)(4,5)(6,22)(7,8)(9,23)(10,11)(13,24)(14,15)(17,18)(20,21)\rangle$$
$$\cong A_4 \times C_2 \text{ regular and normalized by } G$$

N has a normal subgroup P

$$P = \langle (1,4)(2,7)(3,10)(5,12)(6,14)(8,16)(9,17)(11,19)(13,20)(15,22)(18,23)(21,24), (1,5)(2,8)(3,11)(4,12)(6,15)(7,16)(9,18)(10,19)(13,21)(14,22)(17,23)(20,24), (1,2)(3,6)(4,7)(5,8)(9,13)(10,14)(11,15)(12,16)(17,20)(18,21)(19,22)(23,24)\rangle$$

$\cong C_2 \times C_2 \times C_2$ also normalized by G

but G has no normal subgroups of order 8.

However, G has three subgroups of order 8, all isomorphic to D_4, one of which *must* be G' whose fixed field is the same as H_P. The question is, which one?

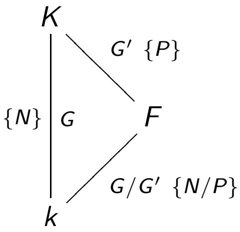
We'll return to this question shortly.

That G' is not normal is not that shocking since F/k does not need to be classically Galois in order to be Hopf-Galois. The question is whether this affects our ability to impose a Hopf-Galois structure on F/k.

We shall begin however with the case where $G' \triangleleft G$ and then look beyond to other situations.

Moreover, we shall consider the ambient symmetric groups in which these (semi-)regular subgroups reside and their relationships with each other.

With this diagram in mind



we define

- $[K : F] = |G'| = |P| = p$
- $[F : k] = [G : G'] = [N : P] = m$
- $[K : k] = |N| = |G| = mp = n$

where p is not necessarily a prime (although perhaps it may be sometimes!).

To simplify the discussion, we shall let $B_X = Perm(X)$ for $|X| = n$ and consider a regular subgroup $G \leq B_X$ as well as a regular subgroup $N \leq B_X$ such that $G \leq Norm_{B_X}(N)$.

We shall also consider $Y \subseteq X$ where $G'_Y \leq B_Y = Perm(Y)$ as a regular subgroup, which gives rise to a semi-regular $G'_X \leq B_X$.

As Y is the set of points on which G'_Y operates, then we have

$$X = X_1 \cup X_2 \cdots \cup X_m$$

where the X_i are the orbits of the action of G'_X, where, WLOG, $X_1 = Y$.

This is not as mysterious as it looks.

Consider $G = \langle\sigma\rangle \cong C_{12}$ and $G' = \langle\sigma^3\rangle$ then

$$\lambda(\sigma) = (1, \sigma, \sigma^2, \ldots, \sigma^{11}) \in Perm(G)$$
$$\lambda(\sigma^3) = (1, \sigma^3, \sigma^6, \sigma^9)(\sigma, \sigma^4, \sigma^7, \sigma^{10})(\sigma^2, \sigma^5, \sigma^8, \sigma^{11}) \in Perm(G)$$
$$\lambda(\sigma^3) = (1, \sigma^3, \sigma^6, \sigma^9) \in Perm(G')$$

and so $X = \{1, \sigma, \ldots, \sigma^{11}\} = X_1 \cup X_2 \cup X_3$ where

$$X_1 = Y = \{1, \sigma^3, \sigma^6, \sigma^9\}$$
$$X_2 = \{\sigma, \sigma^4, \sigma^7, \sigma^{10}\}$$
$$X_3 = \{\sigma^2, \sigma^5, \sigma^8, \sigma^{11}\}$$

and so $\lambda(G')$ is a regular subgroup of $Perm(Y)$ and a semi-regular subgroup of $Perm(X)$.

More formally

Lemma

For G, G', X, and Y as above, $X = X_1 \cup \cdots \cup X_m$ where $X_i = Orb_{G'_X}(x_i)$ for distinct $x_1, \ldots, x_m$ in X and $X_i \cap X_j = \emptyset$ for $i \neq j$ and $|X_i| = |G'|$ and $m = [G : G']$.

Proof.

Since G'_X is a subgroup of a regular permutation group, G, then G'_X is semi-regular, that is it acts fixed point freely. As such, for any $x \in X$, the orbit $Orb_{G'_X}(x)$ contains $|G'_X|$ distinct elements. And as $|G'_X|$ divides $|X|$ then one may simply choose m distinct x_i yielding a partition of X into distinct orbits as in the statement of the result. □

Given this partition of X we can view G'_X as embedded diagonally in

$$Perm(X_1) \times \cdots \times Perm(X_m) \leq B = Perm(X)$$

where the action of G'_X restricted to each $Perm(X_i)$ yields a regular subgroup of that $Perm(X_i)$.

Moreover, given this partition of X, we can view the X_i as 'blocks' with respect to the action of G/G'_X.

Lemma

With $G'_X \triangleleft G$ and X partitioned as above, the quotient group G/G'_X acts on $\{X_1, \ldots, X_m\}$ as a regular permutation group.

Proof.

If $G'_X = \{\sigma_1, \ldots, \sigma_p\}$ then $X_i = Orb_{G'_X}(x_i) = \{\sigma_1(x_i), \ldots, \sigma_p(x_i)\}$ and

$$\gamma\sigma(Orb_{G'_X}(x_i)) = \{\gamma\sigma\sigma_1(x_i), \ldots, \gamma\sigma\sigma_p(x_i)\} = \{\gamma\sigma_1(x_i), \ldots, \gamma\sigma_p(x_i)\}$$

for any $\gamma\sigma \in \gamma G'_X$. But now, since $G'_X \triangleleft G$ we have

$$\begin{aligned}\{\gamma\sigma_1(x_i), \ldots, \gamma\sigma_p(x_i)\} &= \{\sigma_1^{'}\gamma(x_i), \ldots, \sigma_p^{'}\gamma(x_i)\} \\ &= Orb_{G'_X}(\gamma(x_i))\end{aligned}$$

so the action is $\gamma G'_X(Orb_{G'_X}(x_i)) = Orb_{G'_X}(\gamma(x_i))$.

Proof.

This is well defined since $\gamma(Orb_{G'_X}(x)) = Orb_{G'_X}(x)$ if and only if $\gamma \in G'_X$.

The reason for this is that $Orb_{G'_X}(\gamma(x))$ will not contain x unless $\sigma\gamma$ is the identity for some $\sigma \in G'_X$ by regularity of G.

As such $\gamma_1 G'_X(Orb_{G'_X}(x)) = \eta_2 G'_X(Orb_{G'_X}(x))$ if and only if $\gamma_2^{-1}\gamma_1 \in G'_X$. Since G acts regularly on X itself, then given distinct x_i and x_j above, there is some $\gamma \in G$ such that $\gamma(x_i) = x_j$ whence $\gamma G'_X(X_i) = X_j$.

That is, G/G'_X (of order m) acts transitively on $\{X_1, \ldots, X_m\}$ so it must be regular. □

One can pause to observe that G itself may be embedded as a subgroup of

$$(Perm(X_1) \times \cdots \times Perm(X_m)) \rtimes Perm(\{X_1, \ldots, X_m\}) \cong S_p \wr S_m$$

where if $x \in X_i$ and $\tau(x) \in X_j$ then $((\sigma_1, \ldots, \sigma_m), \tau)(x) = \sigma_j(\tau(x))$.

To see why this makes sense, start with the exact sequence $1 \to G' \to G \to G/G' \to 1$ making G an extension of G' by G/G'.

By the Universal Embedding Theorem of Kaloujnine and Krasner [4], if a $1 \to A \to B \to C \to 1$ presents a group B as an extension of A by C then B may be embedded in $A \wr C$ where $A \wr C \cong A^{|C|} \rtimes C$ with C acting by coordinate shift on the 'base group' $A \times \cdots \times A$.

In this setting, the groups in question are embedded as subgroups of the given larger symmetric groups by (semi)regularity. i.e. $G'_X \wr (G/G'_X) \hookrightarrow S_p \wr S_m \hookrightarrow S_{mp}$

Now, there is a parallel partitioning of X which arises due to the subgroup $P \triangleleft N$ where $N \leq B_X$ is regular and P is embedded as a regular subgroup $P_Y \leq B_Y$ and a semi-regular subgreoup $P_X \leq B_X$.

(i.e. just as G' has two forms, $G'_Y \leq B_Y$ regular, and $G'_X \leq B_X$ semi-regular)

As such, one has

$$X = \tilde{X}_1 \cup \tilde{X}_2 \cdots \cup \tilde{X}_m$$

where $\tilde{X}_1 = X_1 = Y$ and $\tilde{X}_i = Orb_{P_x}(\tilde{x}_i)$ for distinct $\tilde{x}_1, \ldots, \tilde{x}_m$ where WLOG $\tilde{x}_1 = x_1$.

The fact that G normalizes P yields an action of G on $\tilde{X}_i$.

Proposition

The group G acts transitively on $\{\tilde{X}_i\}$.

Proof.

Since G normalizes P then G acts on $\{\tilde{X}_1, \ldots, \tilde{X}_m\}$ by

$$\begin{aligned}\gamma(Orb_{P_x}(\tilde{x}_i)) &= \{\gamma\sigma_1(\tilde{x}_i), \ldots, \gamma\sigma_p(\tilde{x}_i)\} \\ &= \{\sigma_1'\gamma(\tilde{x}_i), \ldots, \sigma_p'\gamma(\tilde{x}_i)\} \\ &= Orb_{P_X}(\gamma(\tilde{x}_i))\end{aligned}$$

for any $\gamma \in G$. And since G is a regular subgroup of B_X then it acts transitively on $\{\tilde{x}_1, \ldots, \tilde{x}_m\}$ so therefore it acts transitively on $\{\tilde{X}_1, \ldots, \tilde{X}_m\}$. $\square$

What is the relationship between the two partitions of X into blocks $\{X_i\}$ and $\{\tilde{X}_i\}$?

Corollary

For $G' \triangleleft G$ and $P \triangleleft N$ where G normalizes N and P, we have, after re-indexing if necessary that $\tilde{X}_i = X_i$ for $i \in \{1, \ldots, m\}$.

Proof.

The point is that G operates transitively on $\{X_i\}$ so in particular that $\{X_i\} = Orb_G(X_1)$ but by the proposition $Orb_G(\tilde{X}_1) = \{\tilde{X}_i\}$ but $\tilde{X}_1 = X_1$ so, set-wise, and after renumbering $X_i = \tilde{X}_i$. □

So what happens if G' is not normal in G? Going back to our example earlier, there are (for the given P) three different subgroups G_1', G_2', G_3' of order $|P| = 8$, all isomorphic to D_4 in fact, but to pinpoint which is **the** G', one looks at the orbits of each due to their being semi-regular

$$G_1' \to [1,2,4,5,7,8,12,16],[3,10,11,13,19,20,21,24],[6,9,14,15,17,18,22,23]$$
$$G_2' \to [1,4,5,6,12,14,15,22],[2,3,7,8,23,10,11,16,19],[9,13,17,18,20,21,24]$$
$$G_3' \to [1,4,5,12,13,20,21,24],[2,7,8,9,16,17,18,23],[3,6,10,11,14,15,19,22]$$

And in comparison, the orbits of P are

$\tilde{X}_1 = [1,2,4,5,7,8,12,16]$ $\tilde{X}_2 = [3,6,10,11,14,15,19,22]$ $\tilde{X}_3 = [9,13,17,18,20,21,23,24]$

Thus, the only G' that has an orbit in common (containing 1) with P is G_1' where the shared orbit is $[1,2,4,5,7,8,12,16]$, i.e. $Y = [1,2,4,5,7,8,12,16]$ so that $G_Y' \leq Perm(Y)$ and $P_Y \leq Perm(Y)$ where G_Y' normalizes P_Y to give rise to a Hopf-Galois structure on K/F.

The contrast between

$$X_1 = [1,2,4,5,7,8,12,16] \qquad \tilde{X}_1 = [1,2,4,5,7,8,12,16]$$
$$X_2 = [3,10,11,13,19,20,21,24] \qquad \tilde{X}_2 = [3,6,10,11,14,15,19,22]$$
$$X_3 = [6,9,14,15,17,18,22,23] \qquad \tilde{X}_3 = [9,13,17,18,20,21,23,24]$$

highlights the fact that $G' \triangleleft G$ is necessary in order to have $\{X_i\} = \{\tilde{X}_i\}$.

CURIO: It's not clear whether this is accidental but one can choose *identical* right transversals of G'_x in G and P_X in N, i.e.

$$G/G'_X = N/P_X(coset\ reps.)$$
$$= \{(),$$
$$(1,3,9)(2,6,13)(4,11,23)(5,19,17)(7,15,24)(8,22,20)(12,10,18)(16,14,21)$$
$$(1,9,3)(2,13,6)(4,23,11)(5,17,19)(7,24,15)(8,20,22)(12,18,10)(16,21,14)\}$$

The point though is that with G' a non-normal subgroup of G then G need not act transitively on the $\{X_i\}$ which is indeed the case here, but the N does act transitively on $\{\tilde{X}_i\}$ since $P \triangleleft N$.

$$\tilde{X}_1 = [1,2,4,5,7,8,12,16]$$
$$\tilde{X}_2 = [3,6,10,11,14,15,19,22]$$
$$\tilde{X}_3 = [9,13,17,18,20,21,23,24]$$

As such, G/G'_X can be viewed as acting regularly on $\{\tilde{X}_i\}$ (as N/P clearly does)... even though G/G'_X is not a group. (or is it?)

More on this later.

Back in the setting where $G'_X \triangleleft G$ and $P_X \triangleleft N$ we have that both G/G'_X and N/P_x are regular subgroups of $B_{X/Y} = Perm(\{X_i\})$.

And since G normalizes N and P_X then we consider what it means for N/P_X to be normalized by G/G'_X.

Consider $\gamma g\eta\sigma\gamma^{-1}h \in (\gamma G'_X)(\eta P_X)(\gamma^{-1}G'_X)$ and observe that

$$\begin{aligned}
\gamma\sigma_1\eta\pi\gamma^{-1}\sigma_2(Orb_{P_X}(x_i)) &= \gamma\sigma_1\eta\sigma_2\gamma^{-1}(Orb_{P_X}(x_i)) \\
&= \gamma\sigma_1\eta\pi(Orb_{P_X}(\gamma^{-1}(x_i))) \\
&= \gamma\sigma_1\eta(Orb_{P_X}(\gamma^{-1}(x_i))) \\
&= \gamma\sigma_1(Orb_{P_X}(\eta\gamma^{-1}(x_i))) \\
&= \gamma(Orb_{P_X}(\eta\gamma^{-1}(x_i))) \\
&= Orb_{P_X}(\gamma\eta\gamma^{-1}(x_i)))
\end{aligned}$$

where the last set above is therefore $\gamma\eta\gamma^{-1}P_X(Orb_{P_X}(x_i))$.

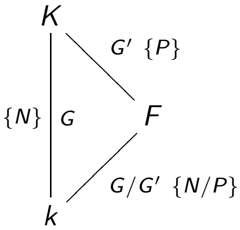
That is

$$(\gamma G')(\eta P)(\gamma^{-1} G') = \gamma\eta\gamma^{-1} P$$

is how G/G' normalizes N/P in $Perm(\{X_i\})$.

One small but useful consequence of this is that $e_G G'$ acts trivially on N/P.

So now, back to



we have K/k is Hopf-Galois with associated group $N \leq B = Perm(G)$ so that $\lambda(G) \leq Norm_B(N))$ where $H = (K[N])^{\lambda(G)}$ is the Hopf algebra which acts and $F \otimes H_P \cong F \otimes (K[P])^{\lambda(G)} \cong (K[P])^{\lambda(G')}$ acts on K/F.

We now have $H_{N/P} = (F[N/P])^{\lambda(G)/\lambda(G')}$ acts on F/k to make it Hopf-Galois since $\lambda(G)/\lambda(G')$ normalizes the regular subgroup N/P, both contained in the same ambient symmetric group.

Moreover, one has an exact sequence of k-Hopf algebras

$$1 \to (K[P])^{\lambda(G)} \to (K[N])^{\lambda(G)} \to (F[N/P])^{\lambda(G)/\lambda(G')} \to 1$$

where $(K[P])^{\lambda(G)} \otimes F$ acts on K/F and the other two terms act on K/k and F/k respectively.

The reason this is exact is that we can rewrite the last term $(F[N/P])^{\lambda(G)/\lambda(G')}$ as

$$\underbrace{(\ \underbrace{(K[N/P])^{\lambda(G')}}_{descend\ \ from\ \ K\ \ to\ \ F}\ )^{\lambda(G)/\lambda(G')}}_{descend\ \ from\ \ F\ \ to\ \ k} = (K[N/P])^{\lambda(G)}$$

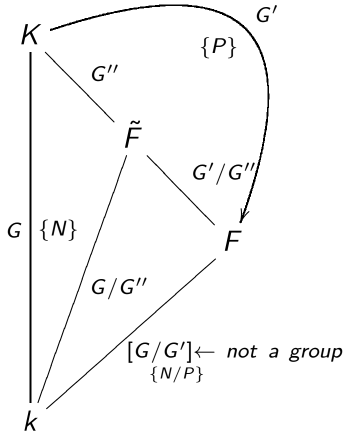
since $(K[N/P])^{\lambda(G')} = F[N/P]$ due to $\lambda(G')$ acting trivially on N/P as observed earlier.

As such, exactness is due to faithful flatness, i.e.

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K[P] & \longrightarrow & K[N] & \longrightarrow & K[N/P] & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \lambda(G)} & & \Big\downarrow{\scriptstyle \lambda(G)} & & \Big\downarrow{\scriptstyle \lambda(G)} & & \\
1 & \longrightarrow & (K[P])^{\lambda(G)} & \longrightarrow & (K[N])^{\lambda(G)} & \longrightarrow & (K[N/P])^{\lambda(G)} & \longrightarrow & 1 \\
 & & & & & & \| & & \\
 & & & & & & (F[N/P])^{\lambda(G)/\lambda(G')} & &
\end{array}
$$

So what if F/k is not classically Galois, i.e. G'_X is not normal in G?

In this case, F is not its own normal closure, but rather $\tilde{F} = K^{G''}$ where, by basic Galois theory, $G'' = \bigcap_{\gamma \in G} gGg^{-1}$ whereby all the intermediate fields diagrammed below are Galois, with the exception of F/k.



On the surface, the fact that G' is not normal in G is not necessarily a problem in the Greither-Pareigis framework since if F/k is to have a Hopf-Galois structure with group N/P then the descent data would come from $Gal(\tilde{F}/k) = G/G''$.

(After all passage to the normal closure of a non-Galois extension is the principal application of the theory in G-P to put Hopf-Galois structures on non-normal extensions.)

This should yield a Hopf algebra $(\tilde{F}[N/P])^{G/G''}$ or, more specifically $(\tilde{F}[N/P])^{\lambda(G/G'')}$ where N/P is viewed as a regular subgroup of $Perm((G/G'')/(G'/G'')) \cong Perm(G/G')$.

So some care must be taken to consider what ambient symmetric groups these groups embedded in, and how they act on each other. (Warning! conjectures ahead)

Can we put a Hopf-Galois structure on F/k (of type N/P) using N and P?

Maybe...

(Yes, I know, "for example" is not proof.)

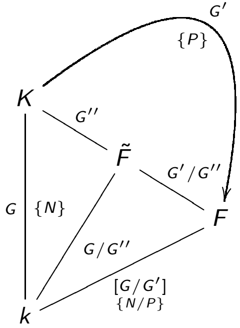
As seen in the earlier example (slide 30), the orbits of G'_X in G, namely $\{X_i\}$ are not acted on transitively by G, but the orbits $\{\tilde{X}_i\}$ of P_X in N are acted on transitively by G, (and of course by N and thus regularly by N/P).

Moreover, as also seen earlier, there is a right transversal of G'_x in G which is not only a group, but acts as a regular permutation group on $\{\tilde{X}_i\}$.

As such, we should view $Perm(G/G')$ as $Perm(\{\tilde{X}_i\})$ and, as also seen in the example earlier, the transversal of G'_X in G is *identical* to that of P in N, and so we have

$$\text{``}G/G' \textit{ normalizes } N/P\text{''}$$

inside the common symmetric group in which both reside.

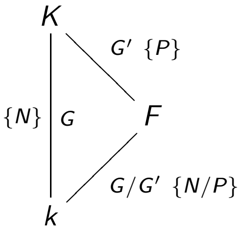


So, getting back to $H_{N/P} = (\tilde{F}[N/P])^{G/G''}$ we can view this descent path from $\tilde{F}$ to k (via G/G'') as being done in two steps

$$\underbrace{(\underbrace{(\tilde{F}[N/P])^{G'/G''}}_{descend\ \ from\ \ \tilde{F}\ \ to\ \ F})^{G/G'}}_{descend\ \ from\ \ F\ \ to\ \ k} = (F[N/P])^{G/G'}$$

which is reasonable since G/G'' acts trivially on N/P so that $(\tilde{F}[N/P])^{G/G''} = \tilde{F}^{G/G''}[N/P] = F[N/P]$.

One 'extremal' case to treat is where $G'' = \{e_G\}$ so that $\tilde{F} = K$ whereby we are back where we started, namely



where any structure on F/k of type N/P is of the form $(K[N/P])^{\lambda(G)}$.

This means that $\lambda(G) \leq Perm(G/G')$ and N/P is embedded as a regular subgroup which must be normalized by $\lambda(G)$.

Here too however, we can view G/G' and N/P as embedded in $Perm(\{\tilde{X}_i\})$ as regular subgroups which normalize each other.

Thank you!

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