A Characteristic-Independent Description of Hopf Orders Using Breuil-Kisin Modules

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Outline

Let:

- R be a complete discrete valuation ring, uniformizing parameter π
- $K = \text{Frac}\, R$
- $k = R/\pi R$, $\text{char}\, k = p > 0$
- H a finite, flat, abelian K-Hopf algebra, rank a power of p.

Objectives.

1. Obtain a classification of R-Hopf orders in K using Breuil-Kisin modules
2. Relate, when possible, the Breuil-Kisin module classifications to other classifications in special cases
3. Decide whether finding such a classification is worth pursuing in general.

Outline

The most general definition I know

Let:

- $W = W(k)$ ring of Witt vectors
- R a complete regular local ring (i.e., $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R =: r$)
- $\mathfrak{S} = W[[u_1, u_2, \ldots, u_r]]$, $\mathfrak{S}_n = \mathfrak{S}/p^n\mathfrak{S}$, $n \geq 1$
- $\sigma : \mathfrak{S} \to \mathfrak{S}$ extend the Frobenius on W: $\sigma(u_i) = u_i^p$
- $\sigma : \mathfrak{S}_n \to \mathfrak{S}_n$ similar
- $E = E(u) \in \mathfrak{S}$ satisfy $E(0) = p$ and $R = \mathfrak{S}/E\mathfrak{S}$
- (for $\mathfrak{M}$ an $\mathfrak{S}$-module) $\mathfrak{S} \otimes_\sigma \mathfrak{M} = \mathfrak{S} \otimes_{\mathfrak{S}} \mathfrak{M}$ with

$$s_1 \otimes_\sigma s_2 m = s_1\sigma(s_2) \otimes_\sigma m; s_1, s_2 \in \mathfrak{S}, m \in \mathfrak{M}.$$

A *Breuil-Kisin module relative to* $\mathfrak{S} \to R$ is a triple $(\mathfrak{M}, \varphi, \psi)$ where:

- $\mathfrak{M}$ is an $\mathfrak{S}$-module which
  - is finitely generated over $\mathfrak{S}$
  - is killed by a power of p
  - has projective dimension at most one
- $\varphi : \mathfrak{M} \to \mathfrak{M}^\sigma$, $\psi : \mathfrak{M}^\sigma \to \mathfrak{M}$ are $\mathfrak{S}$-module maps with

$$\varphi\psi = E \text{ and } \varphi\psi = E.$$

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$$\varphi\psi = E \text{ and } \psi\varphi = E.$$

Theorem (de Jong, case char $R = p$; Kisin, case char $R = 0$)

The category of Breuil-Kisin modules relative to $\mathfrak{S} \to R$ is equivalent to the category of finite, flat, abelian R-Hopf algebras.

The proofs are in terms of Breuil modules and/or group schemes, and not transparent.
We write $\mathfrak{M}$ for $(\mathfrak{M}, \varphi, \psi)$ when the maps are understood.

A very special case

Let:

- $W = W(k)$ ring of Witt vectors, $W_{n-1} = W/p^n W$
- $R = k$ a perfect field, $r = \dim k = 0$ (recall: Krull dimension)
- $\mathfrak{S} = W[[u_1, u_2, \ldots, u_r]] = W,\ \mathfrak{S}_n = \mathfrak{S}/p^n\mathfrak{S} = W_{n-1},\ n \geq 1$
- $\sigma : \mathfrak{S} \to \mathfrak{S}$ is Frobenius on W
- $\sigma : \mathfrak{S}_n \to \mathfrak{S}_n$ similar
- $E = E(u) \in \mathfrak{S}$ satisfy $E(0) = p$ and $R = \mathfrak{S}/E\mathfrak{S}$: $E = p$.
- (for $\mathfrak{M}$ an $\mathfrak{S}$-module) $\mathfrak{S} \otimes_\sigma \mathfrak{M} \cong \mathfrak{M}$.

A *Breuil-Kisin module relative to* $\mathfrak{S} \to k$ is a triple $(\mathfrak{M}, \varphi, \psi)$ where:

- $\mathfrak{M}$ is a finitely generated W=module killed by a power of p
- $F : \mathfrak{M} \to \mathfrak{M}$ is σ-semilinear, $V : \mathfrak{M} \to \mathfrak{M}$ is σ^{-1}-semilinear with

$$FV = VF = p.$$

This should look familiar: classic Dieudonné modules.

The most general definition we'll use from now on

Return to the usual notation: R a complete dvr, etc. So $\dim R = 1$ and

- $\mathfrak{S} = W[[u]],\ \mathfrak{S}_n = W[[u]]/p^n W[[u]],\ n \geq 1$
- $\sigma : \mathfrak{S} \to \mathfrak{S}$ given by $\sigma(u) = u^p$
- $\sigma : \mathfrak{S}_n \to \mathfrak{S}_n$ similar
- $E = E(u) \in \mathfrak{S}$ satisfies $E(0) = p$ and $R = \mathfrak{S}/E\mathfrak{S}$.

A *Breuil-Kisin module relative to* $\mathfrak{S} \to R$ is a triple $(\mathfrak{M}, \varphi, \psi)$ where:

- $\mathfrak{M}$ is an $\mathfrak{S}$-module which
  - is finitely generated over $\mathfrak{S}$
  - is killed by a power of p
  - has no u-torsion
- $\varphi : \mathfrak{M} \to \mathfrak{M}^\sigma,\ \psi : \mathfrak{M}^\sigma \to \mathfrak{M}$ are $\mathfrak{S}$-module maps with

$$\varphi\psi = E \text{ and } \psi\varphi = E.$$

Specialization: primitively generated Hopf algebras

This should also look familiar.

Let $R = k[[T]]$, and suppose H is an R-Hopf algebra generated by its primitive elements.

Then $\psi = 0$.

$E(u) = p$ since $W[[u]]/pW[[u]] \cong R$.

Since $pm = Em = \psi(\varphi(m)) = 0,\ m \in \mathfrak{M}$, it follows that $p\mathfrak{M} = 0$.

Thus, a Breuil-Kisin module is a pair $(\mathfrak{M}, \varphi)$ such that

- $\mathfrak{M}$ is a finitely generated $\mathfrak{S}_1 = k[[u]] \cong R$-module which has no u-torsion (so free, finite rank over R)
- $\varphi : \mathfrak{M} \to \mathfrak{M}^\sigma$ is a linear map (or $\varphi : \mathfrak{M} \to \mathfrak{M}$ is a semilinear map).

These are the Dieudonné modules ($F = \varphi$) from my Exeter talk.

Specialization: characteristic zero DVRs

Let $R/W(k)$ be totally ramified, degree e, Eis. poly. $E_0(u)$.
Let $c_0 = p^{-1}E_0(0) \in W^\times$. $E(u) = c_0^{-1}E_0(u) \equiv c_0^{-1}u^e \bmod p\mathfrak{S}$.
Let $(\mathfrak{M}, \varphi, \psi)$ be a Breuil-Kisin module. So $\varphi\psi = E,\ \psi\varphi = E$.
It can be shown that φ is injective, hence $\psi = \varphi^{-1}E$.
Thus, a Breuil-Kisin module is a pair $(\mathfrak{M}, \varphi)$ where

- $\mathfrak{M}$ is an $\mathfrak{S}$-module which
  - is finitely generated over $\mathfrak{S}$
  - is killed by a power of p
  - has no u-torsion
- $\varphi : \mathfrak{M} \to \mathfrak{M}$ is semilinear such that for all $m \in \mathfrak{M}$ there exist $s_i \in \mathfrak{S},\ x_i \in \mathfrak{M}$ such that

$$Em = \sum_{i=0}^{n} s_i\varphi(x_i).$$

Rank p examples

Pick $0 \leq i \leq e,\ b \in k^\times$.

Let $\mathfrak{M} = \mathfrak{S}_1\mathbf{e} = k[[u]]\mathbf{e}$, and define φ to be the semilinear map with $\varphi(\mathbf{e}) = bu^i\mathbf{e}$.

Then

$$E\mathbf{e} = c_0^{-1}u^e\mathbf{e} = c_0^{-1}b^{-1}u^{e-i}\varphi(\mathbf{e}),$$

and $(\mathfrak{M}, \varphi)$ is a Breuil-Kisin module.

This is a complete classification of rank p Breuil-Kisin modules, though if $b_1 b_2^{-1} \in k^{p-1},\ b_1 b_2 \neq 0$, then their modules are isomorphic.

(Morphisms: $\mathfrak{S}$-module maps, compatible with the φ's and ψ's.)

Outline

Recall $K = \text{Frac}\, R$.

Let H_1, H_2 be R-Hopf algebras (abelian, flat) of the same p-power rank.

We say H_1 and H_2 are *generically isomorphic* if there exists an R-algebra monomorphism $H_1 \to H_2$ which becomes an isomorphism when extended to a K-algebra map $KH_1 \to KH_2$.

So, H_1 generically isomorphic to $H_2 \Rightarrow H_1, H_2$ are R-Hopf orders in the same K-Hopf algebra.

Equivalently, if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are the Breuil-Kisin modules for H_1 and H_2 and there exists a $\mathfrak{S}$-module map $\mathfrak{M}_1 \to \mathfrak{M}_2$ which becomes an isomorphism when inverting u, then H_1, H_2 are generically isomorphic.

Example: characteristic 0

R is a totally ramified extension of $W(k)$ of degree e.

Pick $1 \le i \le e/(p-1)$. Let $\mathfrak{M}_1 = \mathfrak{S}_1\mathbf{e},\ \mathfrak{M}_2 = \mathfrak{S}_1\mathbf{f}$. Let

$$\varphi_1(\mathbf{e}) = u^{i(p-1)}\mathbf{e},\ \varphi_2(\mathbf{f}) = \mathbf{f}$$

and define $\Theta : \mathfrak{M}_1 \to \mathfrak{M}_2$ by $\Theta(\mathbf{e}) = u^i\mathbf{f}$.

Then Θ is a (mono)morphism of Breuil-Kisin modules since

$$\begin{array}{ccccc}
\mathbf{e} & \xrightarrow{\Theta} & u^i\mathbf{f} & = & u^i\mathbf{f} \\
\downarrow{\scriptstyle \varphi_1} & & & & \downarrow{\scriptstyle \varphi_2} \\
u^{i(p-1)}\mathbf{e} & \xrightarrow[\Theta]{} & u^{pi}\mathbf{f} & = & u^{pi}\mathbf{f}
\end{array}$$

Thus, the corresponding Hopf algebras are generically isomorphic.

Example: characteristic p

So $R = k[[T]],\ E(u) = p$. For $j = 1, 2$, let $\mathfrak{M}_1 = \mathfrak{S}_1\mathbf{e},\ \mathfrak{M}_2 = \mathfrak{S}_1\mathbf{f}$.

Note $E\mathfrak{M} = p\mathfrak{M} = 0$.

Pick $i \geq 1$, let

$$\varphi_1(\mathbf{e}) = 0,\ \psi(1 \otimes_\sigma \mathbf{e}) = u^{i(p-1)}\mathbf{e},\ \varphi_2(\mathbf{f}) = 0,\ \psi_2(1 \otimes_\sigma \mathbf{f}) = \mathbf{f},$$

and define $\Theta : \mathfrak{M}_1 \to \mathfrak{M}_2$ by $\Theta(\mathbf{e}) = u^i\mathbf{f}$. Then,

$$\begin{array}{ccccc}
1 \otimes_\sigma \mathbf{e} & \xrightarrow{1\otimes_\sigma\Theta} & 1 \otimes_\sigma u^i\mathbf{f} & = & u^{pi} \otimes_\sigma \mathbf{f} \\
\downarrow{\scriptstyle \psi_1} & & & & \downarrow{\scriptstyle \psi_2} \\
u^{i(p-1)}\mathbf{e} & \xrightarrow[\Theta]{} & u^{i(p-1)}u^i\mathbf{f} & = & u^{pi}\mathbf{f}
\end{array}$$

and the corresponding Hopf algebras are generically isomorphic.

Outline

Killed by $[p]$ is not really killed by $[p]$

Let H be an R-Hopf algebra with Breuil-Kisin module $\mathfrak{M}$.

Recall that $[r] : H \to H,\ r \geq 1$ is defined recursively by

$$[r](h) = \text{mult}([r-1] \otimes 1)\Delta(h).$$

We say H is *killed by p* if $[p]$ is the counit map $\varepsilon : H \to R$.

(Alternatively, the group scheme Spec H is killed by p.)

Example. $RC_p = R\langle\sigma\rangle$ is killed by $[p]$ since $[p](\sigma) = \sigma^p = 1 = \varepsilon(\sigma)$.

H is killed by $[p]$ if and only if $p\mathfrak{M} = 0$.

So, H is killed by $[p]$ iff $\mathfrak{M}$ is a module over $\mathfrak{S}_1 = k[[u]]$, a PID.

Thus $\mathfrak{M}$ is a free $\mathfrak{S}_1$-module, and $\text{rank}_R H = p^{\text{rank}_{\mathfrak{S}_1} \mathfrak{M}}$.

Let $\mathfrak{M} = \oplus_{i=1}^n \mathfrak{S}_1 \mathbf{e}_i = \oplus_{i=1}^n k[[u]]\mathbf{e}_i$.

Pick $A = [a_{i,j}] \in M_n(R)$, and let $\varphi_A : \mathfrak{M} \to \mathfrak{M}$ be the semilinear map

$$\varphi_A(\mathbf{e}_i) = \sum_{j=1}^n a_{j,i}\mathbf{e}_j.$$

That is, $\varphi_A(\mathbf{e}_i) = A\mathbf{e}_i$ when $\mathbf{e}_i$ is represented as a standard basis vector.

Under what conditions is $(\mathfrak{M}, \varphi_A)$ a Breuil-Kisin module?

$\varphi_A(\mathbf{e}_i) = A\mathbf{e}_i$

We require $\{s_{i,j}\} \subset k[[u]]$ such that

$$E\mathbf{e} = \sum_{j=1}^{n} s_{j,i}\varphi_A(\mathbf{e}_j),$$

that is,

$$c_0^{-1}u^e\mathbf{e}_i = \sum_{j=1}^{n}\sum_{\ell=1}^{n} s_{j,i}a_{\ell,j}\mathbf{e}_\ell = \sum_{\ell=1}^{n}\sum_{j=1}^{n} a_{\ell,j}s_{j,i}\mathbf{e}_\ell$$

Let $S = \left[s_{i,j}\right]$. Then we need $c_0^{-1}u^e I = AS$.

Proposition

$(\mathfrak{M}, \varphi_A)$ determines a Breuil-Kisin module structure if and only if $u^e A^{-1} \in M_n(k[[u]])$.

Note that A is necessarily invertible in $M_n(k((u)))$.

$u^e A^{-1} \in M_n(k[[u]])$

Suppose $(\mathfrak{M}, \varphi_A)$ and $(\mathfrak{M}, \varphi_B)$ are Breuil-Kisin module structures on $\mathfrak{M} = \oplus_{i=1}^n k[[u]]\mathbf{e}_i$.

Let Θ be a $k[[u]]$-linear endomorphism of $\mathfrak{M}$, and write

$$\Theta(\mathbf{e}_i) = \sum_{j=1}^{n} \theta_{j,i}\mathbf{e}_i.$$

Let $\Theta \in M_n(k[[u]])$ also represent the matrix $\Theta = [\theta_{i,j}]$.

What are the conditions on Θ to make it a morphism of Breuil-Kisin modules?

Let $A = [a_{i,j}],\ B = [b_{i,j}]$.

We have:

$$\begin{array}{ccccc}
\mathbf{e}_i & \xrightarrow{\Theta} & \displaystyle\sum_{\ell=1}^{n} \theta_{\ell,i}\mathbf{e}_\ell & = & \displaystyle\sum_{\ell=1}^{n} \theta_{\ell,i}\mathbf{e}_\ell \\
\varphi_A \big\downarrow & & & & \big\downarrow \varphi_B \\
\displaystyle\sum_{\ell=1}^{n} a_{\ell,i}\mathbf{e}_\ell & \xrightarrow[\Theta]{} & \displaystyle\sum_{\ell,j=1}^{n} a_{\ell,i}\theta_{j,\ell}\mathbf{e}_j & & \displaystyle\sum_{\ell,j=1}^{n} \theta_{\ell,i}^{p} b_{j,\ell}\mathbf{e}_j
\end{array}$$

So we require

$$\Theta A = B\Theta^{(p)},$$

where $\Theta^{(p)} = [\theta_{i,j}^{p}]$.

Summary

Let $A, B, \Theta \in \mathrm{GL}_n(k((u))) \cap M_n(k[[u]])$, and suppose

1. $u^e A^{-1} \in M_n(k[[u]])$
2. $u^e B^{-1} \in M_n(k[[u]])$
3. $\Theta A = B\Theta^{(p)}$.

Then the Hopf algebras corresponding to $(\mathfrak{M}, \varphi_A), (\mathfrak{M}, \varphi_B)$ are orders in the same K-Hopf algebra.

Note. If $\Theta \in M_n(k[[u]])^\times$, i.e., $\det \Theta \in k[[u]]^\times$, then $(\mathfrak{M}, \varphi_A) \cong (\mathfrak{M}, \varphi_B)$.

Flashback

Exeter talk, 2015.

Overview for all n

- Pick a K-Hopf algebra H, and find the $B \in M_n(K)$ which is used in the construction of its K-Dieudonné module.
- Find $A \in M_n(R)$ such that $\Theta A = B\Theta^{(p)}$ for some $\Theta \in \mathrm{GL}_n(K)$. (One such example: $A = B$, $\Theta = I$.)
- Construct the R-Dieudonné module corresponding to A.
- Construct the R-Hopf algebra H_A corresponding to this Dieudonné module.
- The algebra relations on H_A are given by the matrix A.
- H_A can be viewed as a Hopf order in H using Θ.
- $H_{A_1} = H_{A_2}$ if and only if $\Theta^{-1}\Theta'$ is an invertible matrix in R, where

$$\Theta A_1 = B\Theta^{(p)} \text{ and } \Theta' A_2 = B(\Theta')^{(p)}.$$

Alternatively, $H_{A_1} = H_{A_2}$ if and only if $\Theta' = \Theta U$ for some $U \in M_n(R)^\times$.

Primitively generated Hopf orders in characteristic p.

An example: Hopf orders in KC_p

Fact 1. The Breuil-Kisin module corresponding to RC_p is

$$\mathfrak{M} = k[[u]]\mathbf{e},\ \varphi(\mathbf{e}) = u^e\mathbf{e}.$$

Fact 2. Any Hopf order in KC_p contains RC_p, so we need to construct 1×1 "matrices" B such that

- $u^e B^{-1} \in k[[u]]$
- There exists $\Theta \in k[[u]],\ \Theta \neq 0$ such that $u^e\Theta = \Theta^p B$.

So, we may pick $0 \neq \Theta \in k[[u]]$, and let $B = u^e\Theta^{1-p}$. If

- $u^e B^{-1} = \Theta^{p-1} \in k[[u]]$ (true)
- $u^e\Theta^{1-p} \in k[[u]]$,

then $(k[[u]], \varphi_B)$ gives a Hopf order, different from RC_p iff $\Theta \in uk[[u]]$.

$B = u^e\Theta^{1-p}, \ u^e\Theta^{1-p} \in k[[u]]$

WLOG, let $\Theta = u^i$.

Then
$$B = u^e\Theta^{1-p} = u^{e+i(1-p)} = u^{e-(p-1)i},$$
which is in $k[[u]]$ if and only if $0 \le i \le e/(p-1)$.

The resulting Hopf algebra H_i is the Larson order
$$H_i = R\left[\frac{\sigma - 1}{\pi^i}\right] \subseteq K\langle\sigma\rangle = KC_p.$$

Note. If, e.g., $\Theta = bu^i$ for some $b \in k^\times$ then $B = b^{1-p}u^{e-(p-1)i}$ and the Hopf algebra is the same.

Another example: Hopf orders in KC_p^2

Fact 1. The Breuil-Kisin module corresponding to RC_p^2 is

$$\mathfrak{M} = k[[u]]\mathbf{e}_1 \oplus k[[u]]\mathbf{e}_2, \ \varphi(\mathbf{e}_i) = u^e\mathbf{e}_i, \ i = 1, 2,$$

so the matrix representing φ is $u^e I$.

Fact 2. Any Hopf order in KC_p^2 contains RC_p^2, so we need to construct matrices B such that

- $u^e B^{-1} \in M_n(k[[u]])$
- There exists $\Theta \in \mathrm{GL}_n(k((u))) \cap M_n(k[[u]])$ such that $\Theta u^e I = B\Theta^{(p)}$.

Fact 2. Any Hopf order in KC_p^2 contains RC_p^2, so we need to construct matrices B such that

- $u^e B^{-1} \in M_n(k[[u]])$
- There exists $\Theta \in \mathrm{GL}_n(k((u))) \cap M_n(k[[u]])$ such that $u^e\Theta = B\Theta^{(p)}$.

Alternatively, pick $\Theta \in \mathrm{GL}_n(k((u))) \cap M_n(k[[u]])$ and *define*

$$B = u^e\Theta(\Theta^{(p)})^{-1}.$$

It then suffices to show

$$B = u^e\Theta(\Theta^{(p)})^{-1} \in M_n(k[[u]])$$
$$u^e B^{-1} = \Theta^{(p)}\Theta^{-1} \in M_n(k[[u]]).$$

WLOG, let $\Theta = \begin{bmatrix} u^i & 0 \\ \theta & u^j \end{bmatrix}$, $i, j \geq 0,\ \theta \in k[[u]]$. Then

$$B = u^e\Theta(\Theta^{(p)})^{-1} = \begin{bmatrix} u^{e-(p-1)i} & 0 \\ u^{e-pi}\theta - u^{e-pi-(p-1)j}\theta^p & u^{e-(p-1)j} \end{bmatrix}$$

$$\Theta^{-1}\Theta^{(p)} = \begin{bmatrix} u^{(p-1)i} & 0 \\ u^{-j}\theta^p - u^{(p-1)i-j}\theta & u^{(p-1)j} \end{bmatrix},$$

so we need

$$i, j \leq e/(p-1)$$
$$u^e\theta \equiv u^{e-(p-1)j}\theta^p \bmod u^{pi}k[[u]]$$
$$\theta^p \equiv u^{(p-1)i}\theta \bmod u^j k[[u]].$$

Note that setting $\theta = 0$ gives Larson orders.

Can we make the correspondence explicit?

Can we make the correspondence explicit?

No.

$$\Theta = \begin{bmatrix} u^i & 0 \\ \theta & u^j \end{bmatrix}, \; B = \begin{bmatrix} u^{e-(p-1)i} & 0 \\ u^{e-pi}\theta - u^{e-pi-(p-1)j}\theta^p & u^{e-(p-1)j} \end{bmatrix}$$

What is the Hopf order in $H = K[\langle \sigma_1, \sigma_2 \rangle]$?

$$\varphi_B(\mathbf{e}_1) = u^{e-(p-1)i}\mathbf{e}_1 + (u^{e-pi}\theta - u^{e-pi-(p-1)j}\theta^p)\mathbf{e}_2$$
$$\varphi_B(\mathbf{e}_2) = u^{e-(p-1)j}\mathbf{e}_2$$

Note that $(\mathfrak{M}, \varphi_B)$ is in a short exact sequence of Breuil-Kisin modules

$$0 \to (\mathfrak{S}_1, \varphi_{(e-(p-1)j)I}) \to (\mathfrak{M}, \varphi_B) \to (\mathfrak{S}_1, \varphi_{(e-(p-1)i)I}) \to 0$$

and the Hopf order corresponding to $\mathfrak{M}$ is in $\mathrm{Ext}^1(R[\frac{\sigma_1 - 1}{\pi^i}], R[\frac{\sigma_2 - 1}{\pi^j}])$.

Outline

Another example

Recall $c_0 E \equiv u^e \mod p\mathfrak{S}$.

Fact 1. The Breuil-Kisin module corresponding to $(RC_p^2)^*$ is

$$\mathfrak{M} = k[[u]]\mathbf{e}_1 \oplus k[[u]]\mathbf{e}_2,\ \varphi(\mathbf{e}_i) = c_0\mathbf{e}_i,\ i = 1, 2,$$

so the matrix representing φ is $c_0 I$.

Fact 2. Any Hopf order in $(KC_p^2)^*$ is contained in $(RC_p^2)^*$, so we need to construct matrices A such that

- $u^e A^{-1} \in M_2(k[[u]])$
- $\Theta A = c_0 \Theta^{(p)}$ for some $\Theta \in \mathrm{GL}_2(k[[u]]) \cap M_2(k[[u]])$.

- $u^e A^{-1} \in M_2(k[[u]])$
- $\Theta A = c_0 \Theta^{(p)}$ for some $\Theta \in \mathrm{GL}_2(k[[u]]) \cap M_2(k[[u]])$.

So picking $\Theta \in \mathrm{GL}_2(k[[u]]) \cap M_2(k[[u]])$ and setting $A = c_0 \Theta^{-1} \Theta^{(p)}$ gives a Hopf order if and only if

- $c_0 u^e A^{-1} = u^e (\Theta^{(p)})^{-1} \Theta \in M_2(k[[u]])$
- $\Theta^{-1} \Theta^{(p)} \in M_2(k[[u]])$.

$u^e(\Theta^{(p)})^{-1}\Theta \in M_2(k[[u]]),\ \Theta^{-1}\Theta^{(p)} \in M_2(k[[u]])$

WLOG, let $\Theta = \begin{bmatrix} u^i & 0 \\ \theta & u^j \end{bmatrix}$.

Then

$$u^e(\Theta^{(p)})^{-1}\Theta = \begin{bmatrix} u^{e-(p-1)i} & 0 \\ u^{e-pi}\theta - u^{e-pi-(p-1)j}\theta^p & u^{e-(p-1)j} \end{bmatrix}$$

$$\Theta^{-1}\Theta^{(p)} = \begin{bmatrix} u^{(p-1)i} & 0 \\ u^{-j}\theta^p - u^{(p-1)i-j}\theta & u^{(p-1)j} \end{bmatrix},$$

so we require

$$i, j \leq e/(p-1)$$

$$u^e\theta \equiv u^{e-(p-1)j}\theta^p \bmod u^{pi}k[[u]]$$

$$\theta^p \equiv u^{(p-1)i}\theta \bmod u^j k[[u]].$$

Flashback: Exeter 2015

Hopf orders in $(KC_p^2)^*$ correspond to matrices $\begin{bmatrix} T^i & 0 \\ \theta & T^j \end{bmatrix}$, $\theta \in k[[T]]$ such that

$$T^{-j}\theta^p - T^{(p-1)i-j}\theta \in k[[T]],$$

i.e.,

$$\theta^p \equiv T^{(p-1)i}\theta \text{ mod } T^j k[[T]].$$

Comparison

Characteristic 0	Characteristic p
$\theta^p \equiv u^{(p-1)i}\theta \bmod u^j k[[u]]$ $0 \leq i, j \leq e/(p-1)$ $u^e\theta \equiv u^{e-(p-1)j}\theta^p \bmod u^{pi} k[[u]]$	$\theta^p \equiv T^{(p-1)i}\theta \bmod T^j k[[T]]$ $0 \leq i, j$

So characteristic p behaves like characteristic zero with e sufficiently large.

More generally

Characteristic 0 killed by $[p]$	Characteristic p primitively generated
$A, B \in M_n(k[[u]])$ $\Theta \in \mathrm{GL}_n(k((u))) \cap M_n(k[[u]])$ $\Theta A = B\Theta^{(p)}$ $u^e A^{-1},\ u^e B^{-1} \in M_n(k[[u]])$	$A, B \in M_n(k[[u]])$ $\Theta \in \mathrm{GL}_n(k((u))) \cap M_n(k[[u]])$ $\Theta A = B\Theta^{(p)}$

Characteristic p behaves like characteristic 0
with e sufficiently large, provided A, B invertible.

A singular $\Rightarrow$ the Hopf algebra has nilpotent elements.

Outline

Ideas:

- Establish direct connection between "characteristic 0 killed by $[p]$" and "characteristic p with primitively generated dual".
- Establish connections when the Hopf algebras are not killed by $[p]$.
- Determine connections with/applications to other works
- Decide whether this is worth pursuing.

We discuss each in greater detail.

"char. 0 killed by [p]" & "char. p with prim. gen. dual"

If H is a $k[[T]]$-Hopf algebra with primitively generated dual and Breuil-Kisin module $\mathfrak{M}$, then $\varphi : \mathfrak{M} \to \mathfrak{S} \otimes_\sigma \mathfrak{M}$ is trivial.

Thus, a Breuil-Kisin module is determined by $(\mathfrak{M}, \psi)$ where $\psi : \mathfrak{S} \otimes_\sigma \mathfrak{M} \to \mathfrak{M}$ is any $\mathfrak{S}$-linear map.

We can (presumably) describe ψ in terms of matrices and conjecture:

Characteristic 0 killed by [p]	Characteristic p dual primitively generated
$A, B \in M_n(k[[u]])$ $\Theta \in \mathrm{GL}_n(k((u))) \cap M_n(k[[u]])$ $\Theta A = B\Theta^{(p)}$ $u^e A^{-1},\ u^e B^{-1} \in M_n(k[[u]])$	$A, B \in M_n(k[[u]])$ $\Theta \in \mathrm{GL}_n(k((u))) \cap M_n(k[[u]])$ $\Theta A = B\Theta^{(p)}$

Connections when the Hopf algebras survive [p]

This seems much harder. Suppose $H = KC_{p^2}$. Then

Theorem (K., 2012)

Let $0 \leq j_2 < j_1 \leq e/(p-1)$ *and pick* $f \in k((u))$ *such that*

$$u^{pj_1+(p-1)j_2} f^p - u^{pj_1} f \in k[[u]]$$

$$u^{j_1 - pj_2}(c_0 E(u) - u^e)/p + \left(f^p - u^{-j_2(p-1)} f\right) u^{e+j_1} \in k[[u]]$$

Let $\mathfrak{M} = \mathfrak{S}_2 \mathbf{e}_1 + \mathfrak{S}_2 \mathbf{e}_2$ *with* $p\mathbf{e}_2 = u^{j_1 - j_2}\mathbf{e}_1$. *Let*

$$\varphi_{\mathfrak{M}}(\mathbf{e}_1) = u^{e-(p-1)j_1}\mathbf{e}_1$$

$$\varphi_{\mathfrak{M}}(\mathbf{e}_2) = \left(f^p - u^{-(p-1)j_2} f\right) u^{e+j_1}\mathbf{e}_1 + u^{-(p-1)j_2} E\mathbf{e}_2$$

Then $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ *of this form correspond to the orders in* KC_{p^2}.

$\mathfrak{M} = \mathfrak{S}_2\mathbf{e}_1 + \mathfrak{S}_2\mathbf{e}_2$, $p\mathbf{e}_2 = u^{j_1-j_2}\mathbf{e}_1$.
$\varphi_\mathfrak{M}(\mathbf{e}_1) = u^{e-(p-1)j_1}\mathbf{e}_1$
$\varphi_\mathfrak{M}(\mathbf{e}_2) = \left(f^p - u^{-(p-1)j_2}f\right)u^{e+j_1}\mathbf{e}_1 + u^{-(p-1)j_2}E\mathbf{e}_2$
The sum is not direct, the linear algebra approach seems to fail.

We require $\varphi : \mathfrak{M} \to \mathfrak{S} \otimes_\sigma \mathfrak{M}$, $\psi : \mathfrak{S} \otimes_\sigma \mathfrak{M} \to \mathfrak{M}$ such that $\varphi\psi = E = p$ and $\psi\varphi = E = p$.
In particular, one of φ, ψ need not be trivial.
Any linear algebra approach would require two matrices A_φ, A_ψ which factor pI.

Connections/applications

For example:

Theorem (Childs-Sauerberg, 1998)

Suppose $\mathcal{F} = \Theta^{-1}\mathbb{G}_m^n\Theta$ *is defined over* R*. Suppose that* $f_\Theta = (\Theta^{-1}\Theta^{(p)})^{-1}[p]_\mathcal{F}(\vec{x})$ *is defined over* R *and*

$$(\Theta^{-1}\Theta^{(p)})^{-1}[p]_\mathcal{F}(\vec{x}) \equiv \vec{x}^{(p)} \bmod \pi R.$$

Then $H_\Theta = R[\vec{x}]/(f_\Theta)$ *is a Hopf order in* KC_p^n*.*

What's the connection?

Θ is used above to construct formal groups (generically isomorphic to $\mathbb{G}_m^n$) and isogenies.

Θ in our work is used to construct Hopf algebras, which arise as cokernels of isogenies.

Another example

Let K be a totally ramified extension of $W(k)$, and let L/K be Galois, $\mathrm{Gal}(L/K) = C_{p^n}$.

Theorem (Kohl, 1998)

There are p^{n-1} Hopf Galois structures on L/K.

Since the Hopf algebras are all contained in LC_{p^n} they are necessarily abelian of rank p^n.

The work here could possibly describe their Hopf orders.

This is one of many such results from Byott, Childs, Kohl, etc.

We only require L/K abelian of p-power rank.

Decide whether this is worth pursuing

Is it?

Decide whether this is worth pursuing

Upsides to using Breuil-Kisin modules:

- Easy to understand the theory (at least in characteristic zero)
- Can offer linear algebra solutions to Hopf algebra problems
- Offers insight when transitioning from KG to more general H
- There appears to be some compatibility between Hopf orders in characteristics 0 and p when the Hopf algebras are defined both places
- Offers support to well-known conjectures (e.g. Hopf orders use $n(n+1)/2$ parameters).

Downsides to using Breuil-Kisin modules:

- Working out exact correspondences continues to be very difficult
- Theory only works for H commutative (and cocommutative)
- Without an “initial” or “terminal” Hopf order, some orders may be hard to find (ex: $H = KC_p \otimes KC_p^*$).

Decide whether this is worth pursuing

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- Without an "initial" or "terminal" Hopf order, some orders may be hard to find (ex: $H = KC_p \otimes KC_p^*$)
- You have to sit there for an hour each year and listen to me talk about them.

I wanted 50 slides.

Thank you.