Commutative Hopf-Galois Module Structure of Tame Extensions

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Why study nonclassical HGMS of tame extensions?

- "Better" descriptions of rings of integers in tame Galois extensions of global fields: Martinet's tame quaternionic extensions of $\mathbb{Q}$.
- Descriptions of rings of integers in separable, but non-normal, tame extensions (local or global).
- Uniformity: No known example of a tame H-Galois separable extension L/K of local fields for which $\mathfrak{O}_L$ is not free over $\mathfrak{A}_H$.
- Obvious candidate for the associated order: if $H = E[N]^G$ then $\mathfrak{O}_E[N]^G \subseteq \mathfrak{A}_H$, and there are many examples of equality.

Three theorems (in reverse order)

Theorem

Let L/K be a tame Galois extension of p-adic fields with group G, and let $H = L[N]^G$ be a commutative Hopf algebra giving a Hopf-Galois structure on the extension. Then $\mathfrak{O}_L$ is a free $\mathfrak{O}_L[N]^G$-module.

Three theorems (in reverse order)

- Recall that a separable extension L/K with Galois closure E/K is called *Almost classically Galois* if $\mathrm{Gal}(E/L)$ has a normal complement in $\mathrm{Gal}(E/K)$.

Theorem

Let L/K be a tame almost classically Galois extension of p-adic fields with Galois closure E/K having group G, and let $H = E[N]^G$ be a commutative Hopf algebra giving a Hopf-Galois structure on L/K. Then $\mathfrak{O}_L$ is a free $\mathfrak{O}_E[N]^G$-module.

Three theorems (in reverse order)

Theorem

Let L/K be a tame abelian extension of number fields with group G, and let $H = L[N]^G$ be a commutative Hopf algebra giving a Hopf-Galois structure on the extension. Then $\mathfrak{O}_L$ is a locally free $\mathfrak{O}_L[N]^G$-module.

Sufficient conditions, old and new

- Let L/K be a tamely ramified Galois extension of p-adic fields with group G.

Theorem (PT 2011, 2013)

Suppose at least one of the following conditions is satisfied:

- $p \nmid [L:K]$ and H is commutative;
- The inertia subgroup G_0 acts trivially on N.

Then $\mathfrak{O}_L$ is a free $\mathfrak{O}_L[N]^G$-module.

Theorem

Suppose that N is abelian. The p-part and prime-to-p-part of N are each G-stable. If G_0 acts trivially on the p-part of N, then $\mathfrak{O}_L$ is a free $\mathfrak{O}_L[N]^G$-module.

Induced Hopf-Galois structures

Theorem (Crespo et al. 2016)

Let L/K be a Galois extension of fields with group G and F/K a subextension. Suppose that:

- $\mathrm{Gal}(L/F)$ has a normal complement C in G;
- H_T, H_U, with underlying groups T, U, give Hopf-Galois structures on L/F, F/K respectively.

Then there is a Hopf algebra H with underlying group $T \times U$ giving a Hopf-Galois structure on L/K.

- Say that the Hopf-Galois structure on L/K given by H is *Induced* from those on L/F and F/K.
- In this situation, T, U are G-stable subgroups of $\mathrm{Perm}(G)$.
- Furthermore, the action of the normal complement C on T is trivial.

Conversely...

Theorem (Crespo et al. 2016)

Suppose that

- H gives a Hopf-Galois structure on L/K ;
- the underlying group N is the direct product of two G-stable subgroups T, U;
- $\mathrm{Gal}(L/L^T)$ has a normal complement C in G.

Then:

- there are Hopf algebras H_T, H_U, with underlying groups T, U respectively, giving Hopf-Galois structures on L/L^T, L^T/K respectively;
- the Hopf-Galois structure given on L/K by H is induced from these two Hopf-Galois structures.

- ...and so the action of C on T is trivial.

Putting the pieces together

- Let L/K be a tame Galois extension of p-adic fields with group G, and let $H = L[N]^G$ with N abelian.
- Let T be the Sylow p-subgroup of N; write $N = T \times U$ with $p \nmid |U|$.
- If the action of G_0 on T is trivial, then $\mathfrak{O}_L$ is a free $\mathfrak{O}_L[N]^G$-module.
- If $\mathrm{Gal}(L/L^T)$ has a normal complement C in G then the Hopf-Galois structure given by H on L/K is induced by Hopf-Galois structures on L/L^T and L^T/K respectively, and the action of C on T is trivial.

So...

If $\mathrm{Gal}(L/L^T)$ has a normal complement in G containing G_0, then $\mathfrak{O}_L$ is a free $\mathfrak{O}_L[N]^G$-module.

Normal p-complements for tame Galois extensions

Proposition

Let p^r be the largest power of p that divides $|G|$, and let F/K be a subextension of L/K such that $[L:F]=p^r$. Then $\mathrm{Gal}(L/F)$ has a normal complement in G containing G_0.

Proof.

- G/G_0 is cyclic, so it has a unique normal subgroup of index p^r.
- So G has a unique normal subgroup C of index p^r, containing G_0.
- By the Schur-Zassenhaus theorem, C has a complement in G and the complements of C in G are conjugate.
- But any complement of C in G is a Sylow p-subgroup of G, and these are all conjugate.
- So the complements to C in G are precisely the Sylow p-subgroups of G, and $\mathrm{Gal}(L/F)$ is one of these. □

Towards non-normal extensions: a descent result

Proposition

Let E/K be a Galois extension of p-adic fields with group G and let L/K be a subextension. Let wH_T, H_U give Hopf-Galois structures on E/L, L/K respectively, and let H give the Hopf-Galois structure on E/L induced by these. Suppose that:

- $\mathrm{Gal}(E/L)$ has a normal complement in G;
- E/L is at most tamely ramified;
- $\mathfrak{O}_E$ is a free $\mathfrak{A}_H$-module.

Then $\mathfrak{O}_L$ is a free $\mathfrak{A}_{H_U}$-module.

Almost classically Galois extensions

Theorem

Let L/K be a tame almost classically Galois extension of p-adic fields with Galois closure E/K and let H_U be a commutative Hopf algebra giving a Hopf-Galois structure on L/K. Then $\mathfrak{O}_L$ is a free $\mathfrak{A}_{H_U}$-module.

Proof (Sketch).

- Since L/K is tame, E/L is unramified, hence cyclic. By hypothesis, $\mathrm{Gal}(E/L)$ has a normal complement in G.
- Induce a Hopf-Galois structure on E/K from the structure given by H_U on L/K and the classical structure on E/L. The corresponding Hopf algebra, say H, is commutative.
- By the Galois version of the theorem, $\mathfrak{O}_E$ is a free $\mathfrak{A}_H$-module.
- Now by the descent result, $\mathfrak{O}_L$ is a free $\mathfrak{A}_{H_U}$-module. □

- Thank you for your attention.