The Exercises From Day 3

Wednesday, May 25, 2016

1. Let $H = k[t]/(t^{p^n})$ be monogenic, and let $D_*(H) = Ex$ for some x. We have seen that $F^n x = 0$ and $Vx = fF^r x$ for some $f \in k[F]$ with nonzero constant term. Show that $r > 0$. (bf Hint. It suffices to show that M/FM is not a Dieudonné module if $r = 0$.)

2. As explicitly as possible (which may not be very explicit), write out $\Delta(t)$ for:

   (a) $M = E/E(F^n, V)$.

   (b) $M = E/E(F^n, fF^r - V)$, $1 \leq r < n$.

3. Show that if k contains $\mathbb{F}_{p^{p^2}}$ then
$$E/E(F^3, F^2 - F - V) = E/E(F^3, F - V).$$

4. Show that if $k = \mathbb{F}_{p^2}$ then
$$E/E(F^3, F^2 - F - V) \neq E/E(F^3, F - V).$$

5. Suppose $r = n$. Show that $M \cong M'$ if and only if $r' = n' = n$, where $M = E/E(F^n, fF^r - V)$ and $M' := E/E(F^{n'}, f'F^{r'} - V)$.

6. Suppose $r < n$. Show that if $M \cong M'$ then $n = n'$ and $r = r'$, where $M = E/E(F^n, fF^r - V)$ and $M' := E/E(F^{n'}, f'F^{r'} - V)$.

7. Pick $r > 0$, and suppose $k \subseteq \mathbb{F}_{p^{r+1}}$. For $n > r$ partition the set $\{E/E(F^n, fF^r - V) : f \in \big(k[F]/(F^n)\big)^\times\}$ into isomorphism classes.

8. Count the number of monogenic local-local $\mathbb{F}_p$-Hopf algebras of rank p^n.

9. Suppose k is algebraically closed. For fixed n, r partition the set $\{E/E(F^n, fF^r - V) : f \in \big(k[F]/(F^n)\big)^\times\}$ into isomorphism classes.

10. Continuing with k algebraically closed, count the number of monogenic local-local k-Hopf algebras of rank p^n.

The next four problems refer to the following theorem: Let $M = E/E(F^n, F^r - V)$, $M' = E/E(F^{n'}, F^{r'} - V)$. Then every extension of M by M' is of the form $M_{g,h}$, where $M_{g,h}$ is generated by two elements x, y such that

$$F^n x = gy,\ (F^r - V)x = hy,\ F^{n'} y = 0,\ (F^{r'} - V)y = 0.$$

11. Use these relations above to show that $M_{g,h}$ is killed by a power of F and V (and hence is a Dieudonné module).

12. Let H be the k-Hopf algebra such that $D_*(H) = M_{g,h}$. Show that H is monogenic if and only if $g \in k[F]$ with nonzero constant term.

13. More generally, suppose $g \in F^v f[F] \setminus F^{v+1} k[F]$. Show that $H \cong k[t_1, t_2]/(t_1^{p^{n+n'-v}}, t_2^{p^v})$.

14. Suppose $n = r,\ n' = r',\ g = F^i,\ h = F^j,\ n' \leq i < n,\ j < n'$. Give both the algebra and the coalgebra structure on the bigenic Hopf algebra.

15. Write out all Hopf algebras of height one and rank p^4. Be an explicit as you can, including both the algebra and coalgebra structure.

16. Determine the number of height one Hopf algebras of rank p^6.