The Exercises From Day 1

Monday, May 23, 2016

1. Let $H = R[t]/(t^p),\ \Delta(t) = t \otimes 1 + 1 \otimes t$. Show $P(H) = Rt$.
2. Let $H = R[t]/(t^{p^n}),\ \Delta(t) = t \otimes 1 + 1 \otimes t$. Show $P(H) = Rt + Rt^p + \cdots + Rt^{p^{n-1}}$.
3. Let $H = R[t_1, t_2, \ldots, t_n]/(t_1^p, t_2^p, \ldots, t_n^p),\ \Delta(t_i) = t_i \otimes 1 + 1 \otimes t_i$. Describe $P(H)$ as an R-module.
4. Let $H = R\Gamma$, Γ an abelian p-group (or any finite group). Describe $P(H)$ as an R-module.
5. Let R be a domain, and $H = RC_p^* = \mathrm{Hom}_R(RC_p, R),\ C_p = \langle \sigma \rangle$.
   Let
   $$t = \sum_{i=1}^{p-1} i\epsilon_i,$$
   where $\epsilon_i(\sigma^j) = \delta_{i,j}$.
   Show that $H = R[t]/(t^p - t)$, and $P(H) = Rt$.
6. Prove a subset of the following:
   1. $P(H) \cap R = 0$.
   2. $t, u \in P(H) \Rightarrow t + u \in P(H)$.
   3. $t \in P(H),\ r \in R \Rightarrow rt \in P(H)$.
   4. $P(H)$ is an R-submodule of H.
   5. If R is a PID, then $P(H)$ is free over R.
   6. $t \in P(H) \Rightarrow t^p \in P(H)$.
7. Show that RC_p^2 and RC_{p^2} correspond to the same $R[F]$-module.
8. Show that the $R[F]$-module $(R[F])[X]$ does not correspond to any Hopf algebra H.
9. Let $H = R[t]/(t^{p^2})$ with
   $$\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} t^{pi} \otimes t^{p(p-i)}.$$
   This is a Hopf algebra (not the exercise). Show that H is not primitively generated (yes, the exercise).

10. Use Dieudonné modules to describe $\mathrm{End}(RC_p^*)$

11. Use Dieudonné modules to describe $\mathrm{Aut}(RC_p^*)$.

12. Use Dieudonné modules to describe $\mathrm{End}(R(C_p \times C_p)^*)$.

13. Use Dieudonné modules to describe $\mathrm{Aut}(R(C_p \times C_p)^*)$.

14. Does $\mathrm{Ext}^1_{R[F]}(M, M)$ give all the Hopf algebra extensions? Prove that the answer is no. Hint: consider $H = R[t]/(t^{p^2})$ with

$$\Delta(t) = t \otimes 1 + 1 \otimes t + \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} t^{pi} \otimes t^{p(p-i)}.$$

This H is a Hopf algebra (still not the exercise).

15. Let H be a primitively generated R-Hopf algebra. Prove that $H \otimes_R S$ is a primitively generated S-Hopf algebra and that

$$D_{*,S}(H \otimes_R S) = D_{*,R}(H) \otimes_R S.$$

16. Show that $K[t]/(t^{p^n})$ has no non-trivial L forms for any L.

17. Let $H = \mathbb{F}_p[t_1, t_2]/(t_1^p - t_2, t_2^p - t_1)$. Determine, if possible, the smallest field L such that H and $(KC_{p^2})^*$ are L-forms.

18. Let $M = D_*(H)$ for H a K-Hopf algebra of rank p^n. Suppose F acts freely on M. Show that H and $(K\Gamma)^*$ are K^{sep}-forms for some p-group Γ.

19. Let $M = D_*(H)$ for H a K-Hopf algebra of rank p^n. Suppose $F^r M = 0$ for some $r > 0$. Show that H and $(K\Gamma)^*$ are not K^{sep}-forms for any p-group Γ.

20. Find all Hopf orders in $H = K[t_1, t_2]/(t_1^p, t_2^p)$.

21. Find all Hopf orders in $H = K[t_1, t_2]/(t_1^p, t_2^p - t_2)$.

22. Find all Hopf orders in $H = (KC_p^2)^*$.

23. Determine which of the Hopf orders in $(KC_p^2)^*$ are monogenic.

24. Find all Hopf orders in $H = K[t_1, t_2]/(t_1^p - t_2, t_2^p - t_1)$.

25. Determine which of the Hopf orders in the previous problem are monogenic.