Galois module structure of ideals: some consequences of having a Galois scaffold

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Notation

Let K be a local field with perfect residue field both of prime characteristic, p.

Consider L/K a finite Galois extension of degree p^n, $n \geq 1$ and Gal(L/K)=G.

Also let $\mathfrak{O}_L$ and $\mathfrak{O}_K$ be their respective valuation rings with unique maximal ideals $\mathfrak{P}_L$ and $\mathfrak{P}_K$.

Denote the associated orders of $\mathfrak{P}_L^h$ to be

$$\mathfrak{A}_{L/K}(h) = \{\alpha \in K[G] : \alpha\mathfrak{P}_L^h \subseteq \mathfrak{P}_L^h\} \text{ for } h \in \mathbb{Z}$$

with $\mathfrak{A}_{L/K}$ as associated order of $\mathfrak{O}_L$.

Recall the ramification groups of L/K defined as

$$G_i = \{\sigma \in G : \sigma(x) - x \in \mathfrak{P}_L^{i+1}, \forall x \in \mathfrak{O}_L\} \text{ for } i \in \mathbb{Z}_{\geq -1}.$$

Here we consider totally ramified extensions with ramification numbers $b_1 \leq \cdots \leq b_n$ with $(b_i, p) = 1$.

Galois Scaffold

Assume $b_i \equiv b_n \pmod{p^i}$ for all i. Denote $b \equiv b_n \pmod{p^n}$.
For $t \in \mathbb{S}_{p^n}$ define a map $\mathfrak{a} : \mathbb{S}_{p^n} \to \mathbb{S}_{p^n}$ by $\mathfrak{a}(t) := -b^{-1}t \pmod{p^n}$.
Set $\mathbb{S}_p = \{0, 1, \ldots, p-1\}$ and $\mathbb{S}_{p^n} = \{0, 1, \ldots, p^n - 1\}$. We write $s \in \mathbb{S}_{p^n}$ as $s = \sum_{i=0}^{n-1} s_{(i)} p^i$.

Definition: A *Galois scaffold* on L/K (of tolerance $\mathfrak{T} = \infty$) comprises of :

(1) elements $\lambda_t \in L$ for $t \in \mathbb{Z}$ such that $v_L(\lambda_t) = t$.

(2) $\Psi_i \in K[G]$ for $1 \leq i \leq n$ such that $\Psi_i 1 = 0$ and such that for each i and for each $t \in \mathbb{Z}$ we have

$$\Psi_i \cdot \lambda_t = \begin{cases} \lambda_{t+p^{n-i}b} & \text{if } \mathfrak{a}(t)_{(n-i)} \geq 1, \\ 0 & \text{if } \mathfrak{a}(t)_{(n-i)} = 0. \end{cases}$$

Galois Scaffold 2

For $s \in \mathbb{S}_{p^n}$ write $s \preceq u$ if $s_{(i)} \leq u_{(i)}$ for all $0 \leq i \leq n-1$.
Let $1 \leq b \leq p^n - 1$ and $b - p^n + 1 \leq h \leq b$.
Define

$$d(s) = \left\lfloor \frac{b(s+1) - h}{p^n} \right\rfloor$$

and

$$w(s) = \min\{d(u) - d(u-s) : u \in \mathbb{S}_{p^n}, s \preceq u\}.$$

Theorem (Byott, Childs & Elder,2014, partial)

Suppose L/K has a Galois scaffold. Then the following are true:

(1) $\mathfrak{P}_L^h$ *is free over* $\mathfrak{A}_{L/K}(h)$ *if and only if* $w(s) = d(s)$ *for all* $s \in \mathbb{S}_{p^n}$.

(2) $\mathfrak{A}$ *has* $\mathfrak{O}_K$*-basis* $\{\pi^{-w(s)}\Psi^{(s)} : s \in \mathbb{S}_{p^n}\}$.

Where π is the unifomizer of K and $\Psi^{(s)} = \Psi_n^{s_{(0)}}\Psi_{n-1}^{s_{(1)}} \cdots \Psi_1^{s_{(n-1)}}$.

Ferton in characteristic 0 degree p

Theorem (Ferton, 1972)

Let L/K be a totally ramified extension of degree p with ramification number b and an integer δ witht $0 \leq \delta < p$. Also let b/p have a continued fraction expansion $[0; q_1, q_2, \ldots, q_r]$ of length r, with $q_r \geq 2$.

(i) *If $b = 1$, then $\mathfrak{P}_L^{b-\delta}$ is free over $\mathfrak{A}$ iff $\delta \leq \frac{p-1}{2}$.*

(ii) *If $b > 1$ and $0 \leq \delta \leq b$, then $\mathfrak{P}_L^{b-\delta}$ is free over $\mathfrak{A}$ iff*

  (a) *for even r, $\delta = 0$ or $\delta = q_r$,*

  (b) *for odd r, $\delta \leq q_r/2$.*

(iii) *If $\delta > b > 1$, then $\mathfrak{P}_L^{b-\delta}$ is not free over $\mathfrak{A}$.*

Generalized Ferton for degree p^n

Definition

The integers (b, δ, p^k) with $1 \leq k \leq n$ and $0 \leq \delta < p^k$ are said to satisfy *Ferton condition* if for the continued fraction expansion $\frac{b}{p^k} = [q_0; q_1, q_2, \ldots, q_r]$ of length r, with $q_r \geq 2$ the following holds:

(i) if $b = 1$, then $\delta \leq \frac{p^k - 1}{2}$,

(ii) if $b > 1$, then

  (a) for even r, $\delta = 0$ or $\delta = q_r$,

  (b) for odd r, $\delta \leq q_r/2$.

Theorem

Let L/K be as above of degree p^n. Then $\mathfrak{P}_L^{b-\delta}$ is free **if** *Ferton condition holds for (b, δ, p^k) for at least one value of k.*

Example: $p = 5$, $n = 3$, $b = 33$.

- $\frac{b}{p} = [6; 1, 1, 2]$ $r = 3$ - odd then $\delta = 0, 1$, $h = 32, 33$.
- $\frac{b}{p^2} = [1; 3, 8]$ $r = 2$ - even then $\delta = 8$, $h = 25$.
- $\frac{b}{p^3} = [0; 3, 1, 3, 1, 2, 2]$ $r = 6$ - even then $\delta = 2$, $h = 31$.
- 'sporadic' $h = 29$ and 30.

Duality of values of h

Lemma

Let L/K be a totally ramified extension of degree p^n with a Galois scaffold and let $h + h' \equiv b + 1 \pmod{p^n}$. Then $\mathfrak{P}_L^h$ and $\mathfrak{P}_L^{h'}$ have the same associated orders. More precisely, the two ideals have the same sequence $\{w(s)\}_{s \in \mathbb{S}_{p^n}}$.

Given $h < h'$ we have $d^{h'}(s) \leq d^h(s)$ for all s. There exist s for which $d^{h'}(s) < d^h(s)$ and hence $\mathfrak{P}_L^h$ is not free.

Therefore for every value of b have (at least) $\frac{(p^n-1)}{2} - 1$ non-free ideals.

Special case: $b = p^n - 1$

Lemma

Let L/K be as above and let $b = p^n - 1$. There are precisely $(n+1)$ distinct associated orders for each power of p and $h = 0$. More precisely, if h and h' both satisfy $p^k \mid h$ and $p^{k+1} \nmid h$ for $0 \leq k \leq n-1$, then $\mathfrak{A}_{L/K}(h) = \mathfrak{A}_{L/K}(h')$.

The corresponding $\mathfrak{O}_K$-bases for $\mathfrak{A}_{L/K}(h)$ are as follows:

(i) *when $h = 0$ have $\{1, \pi^{-s}\Psi^{(s)} : s \in \mathbb{S}_{p^n} \backslash \{0\}\}$,*

(ii) *when $p \nmid h$, have $\{1, \pi^{-(s-1)}\Psi^{(s)} : s \in \mathbb{S}_{p^n} \backslash \{0\}\}$,*

(iii) *when $p^k \mid h$ and $p^{k+1} \nmid h$ for some $1 \leq k$, then have*
$$\{1, \pi^{-1}\Psi^{(1)}, \ldots, \pi^{-(p^k-1)}\Psi^{(p^k-1)}, \pi^{-(p^k-1)}\Psi^{(p^k)}, \ldots, \pi^{-(p^k-2)}\Psi^{(p^n-1)}\}.$$

Each associated order has precisely one free ideal $\mathfrak{P}_L^h$ where $h = p^n - p^k$, $(0 \leq k \leq n)$.