Scaffolds and Bondarko Diagrams

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The setup

The aim of this talk is to compare two approaches to local Galois module structure:

- Galois scaffolds (Griff Elder + NB: "our method");
- Bondarko's theory of (semi-)stable extensions, defined via "diagrams".

Notation:

- K: local field of residue characteristic $p > 0$;
- L: totally ramified Galois extension of K of degree p^n;
- $G = \mathrm{Gal}(L/K)$;
- O_L, O_K valuation rings;
- $\mathfrak{P}_L$, $\mathfrak{P}_K$: maximal ideals of these.

For each $h \in \mathbb{Z}$, the **associated order** of $\mathfrak{P}_L^h$ is

$$\mathcal{A}_h = \{\alpha \in K[G] \ : \ \alpha \cdot \mathfrak{P}_L^h \subseteq \mathfrak{P}_L^h\}.$$

This is a subring (O_K-order) in the group algebra $K[G]$, and is the largest subring over which $\mathfrak{P}_L^h$ is a module.

We would like to know when $\mathfrak{P}_L^h$ is **free** as an $\mathcal{A}_h$-module.

In both approaches, the aim is to find a "nice" basis of $K[G]$ whose effect on L can be described using valuations. If we can do this, we can obtain an explicit description of $\mathcal{A}_h$ and a purely numerical condition for $\mathfrak{P}_L^h$ to be free.

Galois scaffolds (simplified version)

A **Galois scaffold** on L/K with **shift** b (with $p \nmid b$) and **tolerance** $\mathfrak{T} > 0$ consists of

- elements $\lambda_t \in L$ for $t \in \mathbb{Z}$ with $v_L(\lambda) = t$;
- elements $\Psi_1, \ldots, \Psi_n \in K[G]$ with $\Psi_i \cdot 1 = 0$;

satisfying the congruence modulo $\mathfrak{P}_L^{t+p^{n-i}b+\mathfrak{T}}$:

$$\Psi_i \cdot \lambda_t \equiv \begin{cases} \lambda_{t+p^{n-i}b} & \text{if } a(t)_{(n-i)} \geq 1 \\ 0 & \text{if } a(t)_{(n-i)} = 0, \end{cases}$$

where

$$a(t) = -b^{-1}t \bmod p^n = a_{(0)} + pa_{(1)} + \cdots + p^{n-1}a_{(n-1)},$$

with $0 \leq a_{(j)} \leq p-1$.
So Ψ_i "typically" increase valuations by $p^{n-i}b$.
Our nice basis of $K[G]$ is $\Psi^{(j)} = \Psi_n^{j_{(0)}} \ldots \Psi_1^{j_{(n-1)}}$.

Bondarko's Approach

We first need to define **diagrams** and the **Bondarko isomorphism**.

Given $\omega \in L \otimes_K L$, pick an expression for ω:

$$\omega = \sum_i x_i \otimes y_i. \tag{1}$$

Note $x \otimes y = kx \otimes k^{-1}y$ for $k \in K^\times$.
We assume (1) is irredundant in the sense that there are no $i \neq j$ with

$$v_L(x_i) - v_L(x_j) = v_L(y_j) - v_L(y_i) \equiv 0 \pmod{p^n}.$$

e.g.

$$\omega = 1 \otimes \mu^3 + \mu^2 \otimes \mu^2 + \mu^2 \otimes \mu^3 + \mu^3 \otimes 1,$$

where $v_L(\mu) = 1$ (and $p^n > 3$).

Bondarko's Approach

Let

$$R(\omega) = \{(v_L(x_i), v_L(y_i))\} \subseteq \frac{\mathbb{Z} \times \mathbb{Z}}{\langle (p^n, -p^n) \rangle}.$$

This depends on the choice of expression (1).

e.g. For $\omega = 1 \otimes \mu^3 + \mu^2 \otimes \mu^2 + \mu^2 \otimes \mu^3 + \mu^3 \otimes 1$,

$$R(\omega) = \{(0,3),\ (2,2),\ (2,3),\ (3,0)\}.$$

Also define the following, which are independent of the choice (1):

- the set of **generators** of ω,

  $$G(\omega) = \text{set of minimal elements of } R(\omega)$$

  where $(u,v) \leq (u',v') \Leftrightarrow u \leq u'$ and $v \leq v'$;
  e.g. $G(\omega) = \{(0,3),\ (2,2),\ (3,0)\}$.

Bondarko's Approach

- the **level** of ω,

$$d(\omega) = \min\{u + v \ : \ (u, v) \in R(\omega)\};$$

  e.g. for

$$\omega = 1 \otimes \mu^3 + \mu^2 \otimes \mu^2 + \mu^2 \otimes \mu^3 + \mu^3 \otimes 1,$$

  we have $d(\omega) = \min\{3, 4, 5\} = 3$.

- the **diagram** of ω,

$$D(\omega) = \{(u', v') \ : \ (u', v') \geq (u, v) \text{ for some } (u, v) \in R(\omega)\};$$

- the **lower diagonal** of ω;

$$N(\omega) = \{(u, v) \in R(\omega) \ : \ u + v = d(\omega)\} \subseteq G(\omega),$$

  e.g.

$$N((\omega) = \{(0, 3), \ (3, 0)\}.$$

Bondarko's Approach

Now we define the **Bondarko isomorphism** $\phi \colon L \otimes L \longrightarrow L[G]$ by

$$\phi(x \otimes y) = \sum_{\sigma \in G} x\sigma(y)\sigma.$$

Inside $L[G]$ we have the K-subspace $K[G]$.

Although ϕ is not an isomorphism of K-algebras, the subspace $\phi^{-1}(K[G])$ is closed under multiplication in $L \otimes L$. So we can define the non-standard multiplication on $K[G]$:

$$\xi * \xi' = \phi\left(\phi^{-1}(\xi)\phi^{-1}(\xi')\right).$$

Write

$$\xi^{*s} = \underbrace{\xi * \cdots * \xi}_{s}.$$

Bondarko's Approach

Bondarko defines L/K to be **semistable** if there is some $\xi \in K[G]$ so that $\omega = \phi^{-1}(\xi) \in L \otimes L$ satisfies

(i) $p \nmid d(\omega)$;

(ii) $|N(\omega)| = 2$.

L/K is **stable** if, furthermore,

(iii) $|G(\omega)| = 2$.

Bondarko's Approach

Bondarko proves:

- If L/K is stable, all ramification numbers are congruent mod p^n to $-d(\omega)$, and ξ^{*s} for $0 \leq s \leq p^{n-1}$ is a "nice" basis.
- Semistable extensions become stable under tamely ramified base change;
- An **abelian** extension is semistable if and only if it comes from a finite subgroup of a 1-dimensional formal group.

For stable L/K, he gives a necessary and sufficient numerical condition for an ideal $\mathfrak{P}_L^h$ to be free over its associated order. (This works for semistable extensions under an additional hypothesis.)

Some initial comparisons

- Bondarko's work came first. The main paper is Bondarko (Contemp. Math., 2002), building on Bondarko (Doc. Math., 2000). For our approach, see Elder (PAMS, 2009), Byott & Elder (JNT, 2013), Byott & Elder (to appear in PAMS), and preprints of Byott, Childs, Elder (2013/2014) on arXiv.
- Bondarko comes close to stating that an ideal can *only* be free in a semistable extension. We make no such claim.
- We give constructions of families of fields with a scaffold. Bondarko gives no explicit examples.
- Bondarko's basis of $K[G]$ uses a single generator ξ, but in the non-standard multiplication. Ours uses n generators $\Psi_1, \ldots, \Psi_n$ (more complicated!) but in the natural multiplication. This enables us to treat questions other than the freeness of $\mathfrak{P}_L^h$ over $\mathcal{A}_h$, e.g. minimal number of generators, embedding dimension of $\mathcal{A}_h$.

Some initial comparisons

- Our approach extends beyond the Galois setting to A-scaffolds, where A is a K-algebra with a suitable action on L. For example A could be a Hopf algebra making the field L into an A-Hopf Galois extension of K. (Here L/K might or might not be normal or separable.)
- Even in the Galois case, our general definition of a scaffold does not tell us how the Ψ_i fit with the Hopf algebra structure on $K[G]$ (beyond being in the augmentation ideal). Bondarko implicitly uses the Hopf algebra structure of $K[G]$ in defining the isomorphism ϕ. This would seem to preclude proving in full generality that "any extension with a scaffold must be semistable".

Generalising the Bondarko Isomorphism

Suppose that H is a finite dimensional cocommutative K-Hopf algebra, and that the field L is an H-Galois extension of K.

We want to define a K-linear isomorphism $\phi\colon L \otimes L \longrightarrow L \otimes H$.

H contains a 1-dimensional subspace of (left and right) **integrals**, i.e. elements θ with $h\theta = \epsilon(h)\theta = \theta h$ for all $h \in H$, where $\epsilon\colon H \longrightarrow K$ is the augmentation.

Pick an integral $\theta \neq 0$, and define

$$\phi(x \otimes y) = \sum_{(\theta)} x(\theta_{(1)} \cdot y) \otimes \theta_{(2)}$$

where

$$\Delta(\theta) = \sum_{(\theta)} \theta_{(1)} \otimes \theta_{(2)}.$$

Generalising the Bondarko Isomorphism

Then ϕ is a K-linear isomorphism (since L is H-Galois) and $\phi^{-1}(1 \otimes H)$ is closed under multiplication (since H is cocommutative).

Note that ϕ depends on the choice of θ, but only up to normalisation by an element of $K^\times$.

In the Galois case $H = K[G]$, we can take

$$\theta = \sum_{\sigma \in G} \sigma,$$

and this gives Bondarko's ϕ.

Example 1: Inseparable Extensions

Let K be a local field of characteristic p, with uniformiser π, and let L be a totally ramified and purely inseparable extension of degree p^n.

Let H be the divided power Hopf algebra of dimension p^n. This has K-basis D_i for $0 \leq i \leq p^n - 1$ where

$$D_i D_j = \binom{i+j}{j} D_{i+j};$$

this is 0 if $i + j \geq p^n$. (Think of D_i as $y^i/i!$.) Then

$$H = K[D_1, D_p. \cdots, D_{p^{n-1}}], \qquad D_{p^j}^p = 0.$$

The comultiplication and augmentation are

$$\Delta(D_r) = \sum_{j=0}^{r} D_j \otimes D_{r-j} \text{ and } \epsilon(D_r) = \delta_{0,r}.$$

Example 1: Inseparable Extensions

Choose $b > 0$ with $p \nmid b$. Then we can write $L = K(x)$ with $v_L(x) = -b$ and $x^{p^n} \in K$. Let H act on L by

$$D_r \cdot x^s = \binom{s}{r} x^{s-r}.$$

This makes L into an H-Galois extension.

Write $\Psi_i = D_{p^{n-i}}$ and $X_i = x^{p^{n-i}}$ for $1 \le i \le n$. Then L has a K-basis

$$X_n^{s_{(0)}} X_{n-1}^{s_{(1)}} \dots X_1^{s_{(n-1)}}, \qquad 0 \le s_{(j)} \le p-1,$$

and

$$\Psi_i \cdot (X_n^{s_{(0)}} X_{n-1}^{s_{(1)}} \dots X_1^{s_{(n-1)}}) = s_{(n-i)} X_n^{s_{(0)}} \dots X_i^{s_{(n-i)}-1} \dots X_1^{s_{(n-1)}}.$$

So Ψ_i "behaves like differentiation with respect to X_i".

Example 1: Inseparable Extensions

For each $t \in \mathbb{Z}$, we need λ_t with $v_L(\lambda) = t$. Let $0 \le a(t) \le p^n - 1$ satisfy $t = -ba(t) + p^n f_t$ with $f_t \in \mathbb{Z}$, and set

$$\lambda_t = \frac{\pi^{f_t} X^{a(t)}}{a(t)_{(0)}! \cdots a(t)_{(n-1)}!} = \pi^{f_t} \prod_{i=1}^{n} \frac{X_i^{a(t)_{(n-i)}}}{a(t)_{(n-i)}!}.$$

Then $v_L(\lambda_t) = t$ and

$$\Psi_i \cdot \lambda_t = \begin{cases} \lambda_{t+p^{n-i}b} & \text{if } a(t)_{(n-i)} \ge 1 \\ 0 & \text{if } a(t)_{(n-i)} = 0. \end{cases}$$

Thus we have a scaffold of tolerance ∞.

Example 1: Inseparable Extensions

What does this look like in Bondarko's set-up?

Claim: $\omega := x \otimes 1 - 1 \otimes x \in L \otimes L$ lies in $\phi^{-1}(H)$, so L/K is stable.

Define ϕ using the integral $\theta = D_{p^n-1}$ with $\Delta(\theta) = \sum_{j=0}^{p^n-1} D_j \otimes D_{p^n-1-j}$. Then

$$\begin{aligned}
\phi(\omega) &= \sum_j x(D_j \cdot 1) D_{p^n-1-j} - \sum_j 1(D_j \cdot x) D_{p^n-1-j} \\
&= x D_{p^n-1} - (x D_{p^n-1} + D_{p^n-2}) \\
&= D_{p^n-2}.
\end{aligned}$$

Moreover, $D_{p^n-2}^{*s} = D^{p^n-1-s}$, so we get essentially the same "nice" basis of H as in our scaffold, but in reverse order.

Example 2: Radical Extensions of Miyata Type

Let K be a local field of characteristic 0 and residue characteristic $p \geq 3$, with absolute ramification index e

Let $a \in K$ with $v_K(a-1) = s$ where $p \nmid s$ and $0 < s < ep/(p-1)$, and take $L = K(\alpha)$ with $\alpha^{p^n} = a$. Then L/K is totally ramified of degree p^n. Define

$$\eta = \alpha - 1.$$

Then $v_L(\eta) = s$.

If K contains a primitive p^nth root of unity ζ then L/K is Galois. Its Galois group is cyclic, generated by σ with $\sigma(\alpha) = \zeta\alpha$. Miyata (1998) studied the Galois module structure of O_L for such L. All the ramification numbers are congruent to $-s$ mod p^n. The group algebra $H = K[G]$ has a basis of primitive idempotents

$$e_i = \frac{1}{p^n} \sum_{j=0}^{p^n-1} \zeta^{-ij} \sigma^j$$

and $e_j \cdot \alpha^k = \delta_{j,k} \alpha^k$ for $0 \leq k \leq p^n - 1$.

Example 2: Radical Extensions of Miyata Type

If $\zeta \notin K$, then L/K is not normal, but we can interpret the σ^j as embeddings $L \hookrightarrow E$, where E is the Galois closure of L/K.

Greither-Pareigis theory describes the Hopf-Galois structures on L/K. Amongst them is an obvious "almost classical" one, in which the Hopf algebra H acting on L has the e_j as a basis. What follows applies in the non-normal case to **this particular** Hopf-Galois structure.

In either case, we can use the (rescaled) Bondarko map ϕ coming from the integral

$$\theta = p^{-n} \sum_j \sigma^j.$$

Then

$$\phi(\alpha^k \otimes \alpha^{-k}) = e_k.$$

In the non-standard multiplication $*$ on H,

$$e_j * e_k = e_{j+k}.$$

Example 2: Radical Extensions of Miyata Type

Let us define

$$\Lambda_k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} e_j, \text{ so } e_j = \sum_k \binom{j}{k} \Lambda_k.$$

Then

$$\begin{aligned} \phi^{-1}(\Lambda_1) &= \phi^{-1}(e_1 - e_0) \\ &= \alpha \otimes \alpha^{-1} - 1 \otimes 1 \\ &= (1+\eta) \otimes (1 - \eta + \eta^2 - \cdots) - 1 \otimes 1 \\ &= \eta \otimes 1 - 1 \otimes \eta + \cdots, \end{aligned}$$

showing that L/K is stable. Also

$$\Lambda_j \Lambda_k = (-1)^{j+k} \sum_h \binom{k}{h} \binom{j+k-h}{k} \Lambda_h,$$

and

$$\Lambda_j \cdot \eta^k = (-1)^{j+k} \sum_h \binom{k}{h} \binom{j+k-h}{k} \eta^h.$$

Example 2: Radical Extensions of Miyata Type

We now want to construct a scaffold. We guess that, as in Example 1, that the scaffold basis elements $\Psi^{(j)}$ should match the Bondarko basis elements Λ_k but in reverse order. So try setting

$$\Psi_r = -\Lambda_{p^n-1-p^{n-r}}.$$

Then

$$\Psi_r \cdot \eta^k = (-1)^k \sum_h \binom{k}{h} \binom{p^n - 1 - p^{n-r} + k - h}{k} \eta^h.$$

The only terms with coefficient not divisible by p are $h = k - p^{n-r}$ and $h = k$. Write $\mathcal{M}$ for the lattice in O_L with O_K-basis η^h for $0 \leq h \leq p^n - 1$. Then

$$\Psi_r \cdot \eta^k \equiv k_{(n-r)} \eta^{k-p^{n-r}} \equiv \binom{k}{p^{n-r}} \eta^{k-p^{n-r}} \pmod{p\mathcal{M} + \eta^k O_L}.$$

So Ψ_r "typically" *decreases* valuations by $p^{n-r}s$.

Example 2: Radical Extensions of Miyata Type

Normalising the η^k appropriately to get elements $\lambda_t \in L$ with $v_L(\lambda_t) = t$, we then get a scaffold of tolerance s, provided that $s \leq e$.

In the Galois case, this means the first ramification number

$$b_1 = \frac{ep}{p-1} - s \geq \frac{e}{p-1}$$

so the ramification of L/K has to be in the "stable" range.

(To apply our general results on scaffolds to read off which ideals are free, we also need to assume $s \geq 2p^n - 1$.)

Remaining Questions

- What happens in the *other* Hopf-Galois structures in the Miyata type extensions (in the Galois and non-Galois cases)?
- We have the class of (near) one-dimensional extensions in characteristic p which have a Galois scaffold. How do they fit into Bondarko's picture?
- How do scaffolds behave under tame base change?