PURELY INSEPARABLE FIELD EXTENSIONS

LINDSAY N. CHILDS

Abstract. We survey some of the research on purely inseparable field extensions, especially between 1968 and 1975 by Sweedler, Gerstenhaber, Rasala, Chase and Waterhouse, and ask about connections between purely inseparable field extensions and Hopf Galois extensions. A bibliography is included.

1. Basic Definitions

Throughout, k be a field of characteristic $p > 0$.

A finite extension K/k of fields is *purely inseparable* if for every α in K, α^{p^m} is in k for some $m \geq 0$. (There is an extensive theory of infinite extensions of fields that I will pass over.)

If K/k is finite and purely inseparable, then $[K : k] = p^e$ for some e. (For if α is in K, then $a = \alpha^{p^m}$ is in k for some minimal m. So $Irr(\alpha, k) = x^{p^m} - a$ for some a in k. Then $[K : k] = [K : k(\alpha)][k(\alpha) : k]$. To show $[K : k] = p^e$ apply induction.)

Examples:

(1) Let $K = k[x]$ with $x^{p^e} = a$, a in k, $a^{1/p}$ not in k. Then $x^{p^e} - a$ is irreducible, so K is a field. K/k has *exponent* e. K is called a *primitive* extension of k.

(2) Let $a_1, \ldots, a_n$ be independent indeterminates over the prime field $\mathbb{F}_p$ and let $k = \mathbb{F}_p(a_1, \ldots, a_n)$. Then

$$k[X_i]/(X_i^{p^{e_i}} - a_i) = k[x_i]$$

is primitive, and

$$K = k[x_1, \ldots, x_n] \cong k[x_1] \otimes_k \ldots \otimes_k k[x_n]$$

is a purely inseparable extension of k. K is a *modular* p. i. extension of k.

(3) ([Sw68], Example 1.1) Let $k = \mathbb{F}_p(a, b, c)$, where a, b, c are independent indeterminates, and let

$$K = k[z, xz - y] \text{ where } z^{p^2} = c, x^p - a, y^p = b.$$

Then K is not modular over k.

Date: May 21, 2013.

More examples later.

2. Prehistory

Purely inseparable field extensions K/k were studied by Teichmuller [Te36], Pickert [Pi49], [Pi59], Jacobson Ja44].

For x in K, the least positive integer e so that x^{p^e} is in k is the *exponent* of e, denoted $e[x : k]$. The largest $e[x : k]$ for all x in K is the exponent of K/k. In this talk all p. i. field extensions will have finite exponent.

For x in K, x is *normal* in K/k if $e[x : k]$ is the exponent of K/k, i. e. $e[x : k] = max\{e[y : k]|y \in K\}$.

A *normal sequence* $x_1, \ldots, x_r$ in K/k is a sequence such that x_1 is normal in K/k and if $K_i = k[x_1, \ldots, x_i]$, then x_{i+1} is not in K_i and x_{i+1} is normal in K/K_i: that is, x_{i+1} has maximal exponent in K/K_i.

A *normal generating sequence* $\{x_1, \ldots, x_r\}$ for K/k is a normal sequence such that $K = K_r = K[x_1, \ldots, x_r]$.

Every purely inseparable extension K/k has a normal generating sequence (Pickert).

3. Derivations

A *Lie algebra* over K is a K-vector space L equipped with a bilinear operation

$$[\ \ ] : L \times L \to L$$

satisfying

- for all $a, b \in K$, $[a, b] = -[b, a]$, hence $[a, a] = 0$ for all a in L
- the Jacobi identity:
$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$
for all a, b, c in K.

Let K/k be a field extension. A *k-derivation* D on K is a k-linear map $K \to K$ such that

$$D(ab) = D(a)b + aD(b)$$

for all a, b in K.

Note that $D(1) = D(1 \cdot 1) = D(1) + D(1)$, hence $D(1) = 0$, hence $D(k) = 0$. The set

$$\{a \in K : D(a) = 0\}$$

is the field of constants of D.

The K-space of k-derivations of K is denoted $\mathcal{D}_k(K)$.

$\mathcal{D}_k(K) \subset End_k(K)$ is a Lie algebra under Lie commutators:

$$[D_1, D_2] = D_1D_2 - D_2D_1$$

and is closed under pth powers:

$$D^k(ab) = \sum \binom{k}{i} (D^i(a)D^{k-i}(b)$$

for all k, so

$$D^p(ab) = D^p(a)b + aD^p(b).$$

Hence $\mathcal{D}_k(K)$ is a *restricted* Lie algebra, because if D is a derivation, then so is $D^p = D \circ \ldots \circ D$ (p factors).

Given a derivation, we can define an algebra homomorphism

$$\Phi : K \to K[t]/(t^2)$$

by

$$\Phi(a) = a + D(a)t.$$

Then $\Phi(ab) = \Phi(a)\Phi(b)$, and $\Phi(a) = a$ iff a is in the field of constants of D.

Example: Let $K = k[x]$ with $x^p = a$ in k. Define a k-derivation $D : K \to K$ by $D(x) = 1$. Then $D(x^r) = rx^{r-1}$ so $D(x^p) = 0 = D(a)$. So D is a k-derivation of K, and $\mathcal{D}_k(K) = KD$. The restricted universal enveloping algebra of $\mathcal{D}_k(K)$ is the exponent one truncated polynomial algebra $K[D]$ since $D^p = D \circ \ldots \circ D = 0$.

Jacobson [Ja44] used derivations in a Galois theory for purely inseparable extensions of exponent one.

Let K/k have exponent one, $K = k[x_1, \ldots, x_m]$, $x_1, \ldots, x_m$ a normal generating sequence.

A *p-basis* of K/k is a set $\{x_1, \ldots, x_r\}$ so that $x^\alpha | \alpha \in \mathbb{F}_p^r$ is linearly independent over k.

Let K/k be purely inseparable and let $\mathcal{D}$ be a restricted Lie algebra of k-derivations of K. Let F be the field of constants of $\mathcal{D}$.

The Galois theory is an inverse correspondence between finite restricted Lie sub-algebras $\mathcal{D}'$ of k-derivations of K and subfields F, $k \subseteq F \subseteq K$, by:

$$\mathcal{D}' \mapsto F, \text{ the field of constants of } \mathcal{D}';$$
$$F \mapsto \mathcal{D}_F(K).$$

If $dim_K(\mathcal{D}') = r$ with K-basis $D_1, \ldots, D_r$, and the field of constants of $\mathcal{D}'$ is F, then K has a p-basis over F with r elements, and $End_F(K)$ has a K-basis consisting of monomials D^α in a K-basis of $\mathcal{D}$, where α runs through $\mathbb{F}_p^r$. (Thus K/F is a H-Hopf Galois extension for $H =$ the restricted universal enveloping algebra of $\mathcal{D}'$.)

Jacobson's exponent one purely inseparable Galois theory motivated a lot of research in the period 1968-75.

4. Sweedler

Sweedler (Annals, 1968, [Sw68]): A finite purely inseparable field extension K/k is *primitive* if $K = k[x]$ with x^{p^e} in k. A *modular* extension is a tensor product of primitive extensions:

$$K = k[x_1] \otimes_k \ldots \otimes_k k[x_r].$$

Every finite purely inseparable field extension K/k of exponent one is modular.

Example (3) illustrates that not all purely inseparable extensions of exponent > 1 are tensor products of primitive extensions.

Sweedler characterized modular extensions in two other ways. First in terms of higher derivations.

Definition. A *higher derivation* of K/k of length n is a series

$$D = (D_0 = I, D_1, \ldots, D_n)$$

of k-homomorphisms from K to K such that for all m,

$$D_m(ab) = \sum_{i=0}^{m} D_i(a)D_{m-i}(b).$$

Let $T = K[t] = K[z]/(z^{n+1})$, a truncated polynomial algebra and define $\Phi : K \to K[t]$ by

$$\Phi(a) = D_0(a) + D_1(a)t + \ldots + D_n(a)t^n.$$

Then the condition that Φ be a k-algebra homomorphism, i. .e. $\Phi(ab) = \Phi(a)\Phi(b)$ translates into

$$D_m(ab) = \sum_{i=0}^{m} D_i(a)D_{m-1}(b)$$

for all m. So $D_0 = I$ and the sequence $\{I, D_1, \ldots, D_n\}$ is a higher derivation of K/k. In particular, D_1 is an (ordinary) derivation of K.

Conversely, every higher derivation of length n yields a k-algebra homomorphism Φ from K to $K[t]$ with $t^{n+1} = 0$.

Gerstenhaber calls Φ an *approximate automorphism* of K/k; Rasala calls Φ a *variation* of K/k in T.

The field of constants of a higher derivation D (or its corresponding approximate automorphism) is the set

$$\{a \in K | \Phi(a) = a\}.$$

Sweedler's second characterization uses linear disjointness.

Definition. Let L, M be subfields of K containing k. Then L and M are *linearly disjoint* over k if every finite set of elements of L that are linearly independent over k are linearly independent over M.

This is a symmetric definition: L and M are linearly disjoint over k iff the multiplication map $L \otimes_k M \to K$ is one-to-one.

Sweedler:

Theorem 4.1. *Let K/k be a purely inseparable field extension of finite exponent. TFAE*

- $K = k[x_1] \otimes_k \ldots \otimes_k k[x_r]$ *(that is, K/k is modular)*
- *k is the field of constants of the set of higher derivations of K/k*
- *K^{p^i} and k are linearly disjoint for all positive i.*

(Mordeson and Vinograde [MV69] have an additional condition related to canonical generators of Pickert.)

Sweedler notes that Example (3): $k = \mathbb{F}_p(a, b, c)$,

$$K = k[z, xz - y] \text{ where } z^{p^2} = c, x^p - a, y^p = b.$$

is not modular because z^p is in the field of constants of all higher derivations of K/k.

Sweedler also proved:

Theorem 4.2. *Let K/k be a purely inseparable field extension of finite exponent n. Then there exists a minimal field $S = S(K/k)$ containing k such that S/k is modular. S/k has exponent n.*

S is the *modular closure* of K/k.

5. Approaches to the proof of the equivalence

There are at least four different proofs of these results in the literature, by Sweedler [Sw68], c.f. [Cj06]; Gerstenhaber [Ge68]; Rasala [Ra71], c.f. [Ka89]; and Waterhouse [Wa75], c.f. [Ka89].

Assume that K/k is finite, purely inseparable of exponent e.

The argument that if $K = k[x_1, \ldots, x_n] \cong k[x_1] \otimes \ldots \otimes k[x_n]$ then k is the field of constants of the set of higher derivations of K/k ("K/k has enough approximate automorphisms") is already in Jacobson's book [Ja64]. First do it for a primitive extension

$$K = k[x] = k[T]/(T^{p^e} - c).$$

Define $\Phi_t : K \to K[t]$, $t^{p^e} = 0$, by $\Phi_t(x) = x + t$. Then

$$\Phi_t(x^{p^e}) = (x+t)^{p^e} = x^{p^e} = c = \Phi(c),$$

so Φ is a well-defined k-algebra homomorphism.

Let $\sum_{i=0}^{d} a_i x^i$ be in K with $a_d \neq 0$. Then

$$\Phi_t(\sum_{i=0}^{d} a_i x^i)$$

has leading term $a_d t^d$ as a polynomial in t. So $\Phi(f) = f$ implies $d = 0$ and f is in k.

It follows easily that if $K = k[x_1] \otimes_k \ldots \otimes_k k[x_n]$, then K/k has enough approximate automorphisms.

Here is Sweedler's argument that if K/k has enough approximate automorphisms, then k and K^{p^n} are linearly disjoint for all $n > 0$.

First we observe that if Φ_t is an approximate automorphism of K/k, then Φ yields by restriction an approximate automorphism of $K^{p^n}/K^{p^n} \cap k$. For

$$\Phi_t(a^p) = \Phi_t(a)^p \pmod{t^{m+1}}.$$

Hence $\phi_j(a^p) = 0$ if p does not divide j, and $\phi_{pi}(a^p) = (\phi_i(a))^p$ for $pi \leq m$. So for all $n > 0$,

$$\Phi_t : K^{p^n} \to K^{p^n}[t],$$

hence ϕ_j maps K^{p^n} to K^{p^n} for all ϕ_j.

Also, if a is in k, b in K, then since $\Phi_t(a) = a$, we have $\phi_j(ab) = a\phi_j(b)$.

Now, suppose k and K^{p^n} are not linearly disjoint. Then there is a finite set $\{\gamma_i\}$ of elements of k, linearly independent over $k \cap K^{p^n}$ but linearly dependent over K^{p^n}. Hence there are elements $\{a_i\}$ in K^{p^n}, so that

$$0 = \gamma_1 a_1 + \ldots + \gamma_m a_m.$$

Assume given such a dependence relation with m minimal. By dividing by a_1 we may assume $a_1 = 1$. Since the γ_i are linearly dependent over $k \cap K^{p^n}$, we may assume that a_2 is not in $k \cap K^{p^n}$, hence there is an approximate automorphism Φ_t of K/k such that $\Phi_t(a_2) \neq a_2$, hence a coefficient ϕ_j of Φ_t so that $\Phi_j(a_2) \neq 0$.

Apply ϕ_j to the dependence relation

$$0 = \gamma_1 a_1 + \ldots + \gamma_m a_m.$$

Since γ_1 is in k, we have $\phi_j(\gamma_1) = \gamma_1 \phi_j(1) = 0$, and

$$0 = \gamma_2 \phi_j(a_2) + \ldots + \gamma_m \phi_j(a_m).$$

This is a shorter dependence relation over $k \cap K^{p^n}$, a contradiction. Hence k and K^{p^n} are linearly disjoint over their intersection.

The difficult part of the equivalent conditions of Sweedler's characterization of a modular extension K/k is getting from linear disjointness, or from enough approximate automorphisms, to the description of K as a tensor product of primitive extensions.

- Gerstenhaber gets from enough approximate automorphisms to modularity by an induction argument, proving a limited version of linear disjointness on the way.
- Sweedler does it by setting up and working with a lower triangular matrix of generating elements of K/k.
- Waterhouse defines a purely inseparable extension K/k to be modular if k and K^{p^n} are linearly disjoint for all n. This allows him to extend the notion of modular to infinite extensions with infinite exponent. He obtains as a special case the result that if $K^{p^n} \subseteq k$ for some n, that is, if K/k has finite exponent, then K is a tensor product of primitive extensions. But there exists a countable dimensional modular extension K/k which is not a tensor product of primitive extensions.

As for Sweedler's theorem about the modular closure of a finite exponent purely inseparable extension K/k, Rasala approaches this as follows:

Pickert ([Pi50]): Let $\{x_1, \dots, x_r\}$ be a normal generating sequence for K/k, with Let $K_i = k[x_1, \dots, x_i]$ and let $p^{e_i} = q_i = [K_{i+1} : K_i]$. Then

$$x^{q_i} \in k[x_1^{q_i}, \dots, x_{i-1}^{q_i}].$$

Hence

$$x_i^{q_i} = \sum_{\alpha \in I_i} a_{i,\alpha} x^{q_i \alpha}$$

for I_i a multi-index set and $a_{i,\alpha}$ in k.

This was a starting point for Rasala [Ra71]:

In some algebraic closure of k, let $d_{i,\alpha}$ be the q^i th root of $a_{i,\alpha}$. Let

$$S(K/k) = K[d_{i,\alpha}]$$

for all $1 \leq i \leq r$ and all α. Then S is a *splitting field* for K/k: that is, $A = S \otimes_k K$ is a simple truncated polynomial algebra.

To show that A is a STPA, define u_i in

$$A = S \otimes_k K = K[d_{i,\alpha}] \otimes_k k[x_1^{q_1}, \dots, x_r^{q_r}]$$

by

$$u_i = x_i - \sum_{\alpha \in I_i} d_{i,\alpha} x^\alpha.$$

Then $u_i^{q_i} = 0$ and

$$A = S[u_1, \ldots, u_r].$$

Rasala and Sweedler showed that if you begin with an exponent n purely inseparable extension K/k and take its minimal splitting field $S_1 = S(K, k)$ of K/k, it will be exponent n but may not be modular over k. (Example (6) below.) But iterate:
let $S_2 = S(S_1, k), S_3 = S(S_2, k)$, etc.,
then the chain stops after a finite number of steps to give a field S so that S/k has exponent n and is modular.

If $K = k[x_1, \ldots, x_r]$ is modular over k with $e(x_i, k) = e_i$, then $K \otimes_k K = K[u_1, \ldots u_r]$ where $u_i^{e_i} = 0$. Thus $K \otimes_k K$ is a simple truncated polynomial algebra.

Waterhouse observes that for a finite modular extension K/k,

$$K \otimes K \cong A$$

where the simple truncated polynomial algebra A is isomorphic to $K[G]$ for a finite abelian p-group.

He extends this idea to show that if G is any p-primary abelian group, finite or not, then there exists a modular extension K/k (modular in the sense of linear disjointness) so that $K \otimes_k K \cong K[G]$. But he has an example of a modular extension K/k such that $\overline{k} \otimes_k K$ is not isomorphic to $\overline{k}[G]$ for any abelian group.

Examples:

(4) ([Sw68], Example 1.2) Let $k = \mathbb{F}_p(a, b, c)$, where a, b, c are independent indeterminates, and let

$$K = k[z, xz - y, x^p, y^p] \text{ where } z^{p^2} = c, x^{p^2} - a, y^{p^2} = b.$$

Then

$$K = k[z] \otimes_k k[xz + y] \otimes_k k[y^p]$$

so K is modular over k. But K is not modular over $K[x^p, y^p]$.

(5) ([Ra71], p. 427) Let $k = \mathbb{F}_p(a, b, c)$ where a, b, c are independent indeterminates, and let

$$K = k[x, w] \text{ where } x^{p^2} = a, w^p = b + cx^p.$$

Then K is contained in the modular extension $S = k[x, y, z]$ where $y^p = b, z^p = c$.

(6) ([Ra71], p. 427). Let $k = \mathbb{F}_p(a, b, c, d)$ where a, b, c, d are independent indeterminates, and let

$$L = k[u, v], M = k[z, w]$$

where $u^{p^3} = d, v^{p^2} = a+(b^p+c^p a)u^{p^2}$; and $z^{p^2} = a, w^{p^2} = b^p+c^p a$. Then the splitting field of L/k is $N = k[u, z, w]$; the splitting field of M/k is $k[z, b^{1/p}, c^{1/p}]$ and the splitting field of N is $S = k[u, z, b^{1/p}, c^{1/p}]$. which is modular over k. So L/k requires two iterations of a splitting field before one reaches a modular extension of k containing L.

An exposition of the papers of Rasala and Waterhouse may be found in [Ka89].

There are a number of papers by Davis, Devaney, Haddix, Heerema, Mordeson, Tucker, Vinograde and Zerla on aspects of the structure of purely inseparable field extensions, both finite and infinite. On the latter, there is Mordeson and Vinograde's LNM [MV70] and Waterhouse [Wa75]. Mordeson and Deveney [DM79], [Mo81] have commentaries on aspects of Waterhouse's treatment of infinite purely inseparable extensions.

6. Galois theory

Davis [Da69] seems to have the first Galois theory extending Jacobson, for exponent 2 field extensions.

Heerema [He71]: For K of characteristic p, let A be the subgroup of $Aut_K(K[x]/(x^{p^e+1})$ consisting of f with $f(x) = x$. Then Heerema developed a Galois theory involving subgroups of A and subfields F of K such that K/F is modular of bounded exponent. The theory was further developed by Mordeson [Mo75].

Gerstenhaber and Zaromp [GZ70] developed a Galois theory using higher derivations. Another version was developed by Heerema and Devaney [HD74]. There is an exposition of these theories in [DM96].

I haven't looked at any of that work in detail.

7. Hopf Galois structures

Chase and Sweedler [CS68] generalized the notion of Galois extension of fields to that of a Hopf Galois extension, or of a Galois object relative to a Hopf algebra. For k a field and H a finite cocommutative k-Hopf algebra, a field extension K/k is an H-module algebra if

$$h(ab) = \Delta(h)(a \otimes b)$$

for all a, b in K and h in H, and $h(1) = \epsilon(h)(1)$. Then K/k is an H-Galois extension if the map

$$j : K \otimes_k H \to End_k(K), j(a \otimes h)(b) = ah(b)$$

is bijective. (The theory works to some degree much more generally than for field extensions.)

Chase and Sweedler developed a Galois theory for Hopf Galois extensions of fields. (Actually they developed it for $H*$-Galois objects.) One motivation was to extend Jacobson's exponent one Galois theory. In the exponent one case, K/k is a H-Galois extension for H the restricted universal enveloping algebra of $\mathcal{D}(K/k)$, so the Chase-Sweedler theory apparently recovers Jacobson's theory (c.f. [Ch71]). But for modular extensions K/k of exponent > 1, there appears to be no "natural" k-Hopf algebra acting on K that makes K/k into a Hopf Galois extension to which the Chase-Sweedler setup can apply–c. f. [Ho74], pp. 222ff.

Explicit descriptions of Hopf Galois structures on purely inseparable field extensions are scattered and cryptic. Here is what I found:

(1) [GP87, p. 240]: "Many purely inseparable extensions are H-Galois [CS69]".

(2) [Ch69, p. 16]: Let $A = k[z]/(z^{p^n}) = k[t]$ with $t^{p^n} = 0$. Then A has, in a unique way, a Hopf algebra structure with t primitive:

$$\Delta(t) = t \otimes 1 + 1 \otimes t, \epsilon(t) = 0, \lambda(t) = -t.$$

Let $S = k[x]$ be a primitive purely inseparable field extension of k of exponent n. Then S is a Galois A-object with

$$\alpha : S \to S \otimes A$$

by

$$af(x) = x \otimes 1 + 1 \otimes z.$$

The same example, with $A^* = H$ acting on S/k, is in Sweedler's book [Sw69, p. 215] as an "Example-Exercise".

(4) In [Ch74], Section 5, Chase proves:

Theorem 7.1 (Chase's Proposition 5.2). *The conditions below are equivalent for any finite field extension K/k:*

- *K/k is a principal homogeneous space (PHS) for some truncated k-group scheme G;*
- *K/k is a principal homogeneous space (PHS) for some commutative truncated k-group scheme G;*
- *K/k is purely inseparable and modular.*

Part of proof: Suppose K/k is purely inseparable and modular, so that $K = k[x_1, \ldots, x_n]$ with only the relations $x_i^{p^{e_i}} = a_i$ for some $a_1, \ldots, a_n$ in k and $e_1, \ldots, e_n > 0$. Let

$$C = k[t_1, \ldots, t_n], t_i^{p^{e_i}} = 0.$$

Then C is a truncated polynomial algebra, and is a Hopf k-algebra by defining Δ, ϵ and λ by

$$\Delta(t_i) = t_i \otimes 1 + 1 \otimes t_i$$
$$\epsilon(t_i) = 0$$
$$\lambda(t_i) = -t_i$$

for $1 \leq i \leq n$.

Then C is a commutative and cocommutative Hopf k-algebra, hence $G = Spec(C)$ is a commutative k-group scheme. Then K/k is a PHS for G with action $\alpha : K \to K \otimes C$ defined by

$$\alpha(x_i) = x_i \otimes 1 + 1 \otimes t_i.$$

For K/k finite and modular, Chase defines the truncated automorphism scheme $G_t(K/k)$ by

$$G_t(K/k)(T) = Aut_T(K \otimes_k T).$$

for any truncated polynomial k-algebra T. Given $G = Spec(C)$ with C as just described,

$$G_t(K/k)(T) = Alg_k(C, K \otimes T).$$

If $K = k[\alpha_1, \ldots, \alpha_s]$ with $\alpha_j^{p^e_j} = 0$, and $n = [K : k]$, then $G_t(K/k)$ is represented by the truncated polynomial algebra

$$P(K/k) = k[t_{i,j}], i = 1, \ldots, n, j = 1, \ldots, s$$

with

$$t_{i,j}^{p^{e_j}} = 0 :$$

that is,

$$G_t(K/k), T) = Alg_k((P(K/k), T).$$

The action of $G_t(K/k)$ on K/k is induced by the coaction map

$$\theta : K \to K \otimes_k P(K/k),$$

given by

$$\theta(\alpha_j) = \alpha_j \otimes 1 + \sum_{i=1}^{n} \beta_i \otimes t_{i,j}$$

where $\{\beta 1, \dots, \beta_n\}$ is some k-basis of K.

Chase shows that $G_t(K/k)$ is independent of the choice of $G = Spec(C)$ making K/k into a PHS.

[Ch74, p. 471]: "Scrutiny of the simplest examples shows that a modular extension can be a PHS for many different truncated group schemes G." But if G is a truncated k-group scheme acting on K/k, then there is a unique morphism $G \to G_t(K/k)$ of k-group schemes that preserves this action.

There are mysterious potential connections between Chase's $G_t(K/k)$, Heerema's subgroup of $Aut_K(K[x]/(x^{p^e+1})$, and Weisfield's result [We65] that K/k is modular iff k is the constant field of a single approximate automorphism.

8. Automorphism schemes

Chase's work relates to the automorphism scheme of a finite field extension:

$$Aut_{K/k}(S) = \{S - \text{algebra automorphisms of } K \otimes_k S\}.$$

There are a few papers in the literature that study automorphism schemes: Begueri [Be69], Shatz [Sh69], Waterhouse [Wa71], Chase [Ch72], Chase [Ch74], Sancho de Salas [SS79] between 1969 and 1979. I didn't have access to the papers of Shatz and Begueri while preparing this, have a fuzzy recollection that for K/k separable with $[K : k] = n$, then the symmetric group S_n shows up somehow.

9. Final remarks

(1) The study of purely inseparable field extensions seems to have been very fashionable in the period between 1968 and 1981, but then essentially stopped. Other than expositions by Karpilovsky [Ka89] and Devaney/Mordeson [DM95], there is almost nothing in the literature– nearly all the citations to the papers in the 1960's and 70's end by around 1981.

(2) From the point of view of Galois module theory, it would be of interest to see if there is an analogue of the Greither-Pareigis correspondence for finite modular purely inseparable Galois extension K/k. Chase's work hints at the possibility of such an analogue.

One could ask: what are the Hopf Galois structures on a simple truncated polynomial K-algebra A? Can one identify the Hopf Galois structures on A/K that are lifts of Hopf Galois structures on K/k?

Is that a reasonable way to think about the question of determining the Hopf Galois structures on a finite modular purely inseparable field extension?

(3) If one could determine Hopf Galois structures on modular p. i. extensions of local function fields, one might then be interested in the Galois module structure of the corresponding extensions of valuation rings. My brief, uninformed search of the literature found virtually nothing on that. For example, Michael Rosen's book "Number Theory in Function Fields" entirely avoids purely inseparable extensions.

References

[Be69] L. Begueri, Scheme d'automorphismes et application a l'etude d'extensions finites radicielle, Bull. Sci. Math. 93 (1969), 89–111.

[Ch69] S. U. Chase, Galois objects and extensions of Hopf algebras, in Category Theory, Homology Theory and their Applications II, Springer LNM 92 (1969), 10–31.

[Ch71] S. U. Chase, On inseparable Galois theory, Bull. Amer. Math. Soc. 77 (1971), 413–417.

[Ch72] S. U. Chase, On the automorphism scheme of a purely inseparable field extension, in Ring Theory (R. Gordon, ed.), Academic Press, 1972.

[Ch74] S. U. Chase, Infinitesimal group scheme actions on finite field extensions, Amer. J. Math. 98 (1974), 441–480.

[CS69] S. U. Chase, M. E. Sweedler, Hopf Algebras and Galois Theory, Springer LNM 97 (1969).

[Ch06] Chellali, Mustapha; Fliouet, El Hassane, Sur la tour des cltures modulaires. [On the tower of modular closures] , An. Stiint. Univ. "Ovidius" Constanta Ser. Mat. 14 (2006), no. 1, 4566.

[Da69] R. L. Davis, A Galois theory for a class of purely inseparable exponent two field extensions, Bull. Amer. Math. Soc. 75 (1969), 1001–1004.

[Da73] R. L. Davis, Higher derivations and field extensions, Trans. Amer. Math. Soc. 180 (1973), 47–52.

[De74] J. Deveney, An intermediate theory for a purely inseparable Galois theory, Trans. Amer. Math Soc. 198 (1974), 287–295.

[DM79] J. Deveney, J. Mordeson, Inseparable extensions and primary abelian groups, Arch. Math. (Basel) 33 (1979-80), 538–545.

[DM95] J. Deveney, J. Mordeson, Higher derivation Galois theory of inseparable extensions, Handbook of algebra, vol. 1 , North-Holland, 1995, 187–220.

[Ge64] M. Gerstenhaber, On the Galois theory of inseparable extensions, Bull. Amer. Math. Soc. 70 (1964), 561–566.

[Ge68] M. Gerstenhaber, On modular field extensions, J. Algebra 10 (1968), 478–484.

[GZ70] M. Gerstenhaber, A. Zaromp, On the Galois theory of purely inseparable field extensions, Bull. Amer. Math. Soc. 76 (1970), 1011–1014.

[GP87] G. Greither, B. Pareigis, Hopf Galois theory for separable field extensions, J. Algebra 106 (1987), 239–258.

[He71] N. Heerema, A Galois theory for inseparable field extensions, Trans. Amer. Math. Soc. 154 (1971), 193–200.

[He75] N. Heerema, A modular Galois theory, Conf. on Commutative Algebra–1975, Queen's Univ., Queen's Papers on Pure and Appl. Math. 42 (1975) .

[HD74] N. Heerema, J. Deveney, Galois theory for fields K/k finitely generated, Trans. Amer. Math. Soc. 189 (1974), 263–274.

[Ho74] R. T. Hoobler, Purely inseparable Galois theory, in Ring Theory, Proceedings of the Oklahoma Confernce, Dekker, 1974, 207–240.

[Ja44] N. Jacobson, Galois theory of purely inseparable fields of exponent one, Amer. J. Math. 66 (1944), 645–648.

[Ja64] N. Jacobson, Lectures in Abstract Algebra III, Theory of Fields and Galois Theory, Van Nostrand, 1964.

[Ka89] G. Karpilovsky, Derivations, and Purely inseparable extensions, Chapters 4 and 5 of Topics in Field Theory, North-Holland, 1989.

[Ki83] T. Kikuchi, On Galois correspondence between intermediate fields and closed derivation subalgebras, J. Math. Kyoto Univ. 23 (1983), 281–287.

[Ki73] Kime, Linda Almgren Allowable diagrams for purely inseparable field extensions. Proc. Amer. Math. Soc. 41 (1973), 389-393.

[Mo75] J. Mordeson, On a Galois theory for inseparable field extensions, Trans. Amer. Math. Soc. (1975), 337–347.

[M081] J. Mordeson, Modular extensions and abelian groups, Arch. Math.(Basel) 36 (1981), 13-20.

[MV69] J. Mordeson, B. Vinograde, Tensor products of simple pure inseparable field extensions, Proc. Amer. Math. Soc. 20 (1969), 209–212.

[MV70] J. Mordeson, B. Vinograde, Structure of arbitrary purely inseparable extension fields, Springer LNM 173 (1970).

[Na70] Y. Nakai, High order derivations I, Osaka J. Math. 7 (1970), 1-27.

[Pi49] G. Pickert, Math. Z. 52(1949), 81–135.

[Pi50] G. Pickert, Eine normalform fur endliche rein-inseparable Korpererweiterungen, Math. Z. 53 (1950), 133–135.

[Ra71] R. Rasala, Inseparable splitting theory, Trans. Amer Math. Soc. 162 91971), 411–448.

[SS79] P. J. Sancho de Salas, Automorphism scheme of a finite field extension, Trans. Amer. Math. Soc. 352 (1979), 595–608.

[Sh69] S. Shatz, Galois theory, Springer LNM 86 (1969), 146–158.

[Sw68] M. Sweedler, Structure of inseparable extensions, Annals of Math. 87 (1968), 401-410; 89 (1969), 206-207; cf. M. Chellali, Erratum on: "Structure of inseparable extensions" [Ann. of Math. (2) 87 (1968), 401410; MR0223343] by M. E. Sweedler, Int. Math. Forum 2 (2007), 65-68, 32693272.

[Sw69] M. Sweedler, Hopf Algebras, Benjamin, 1969.
[Te36] O. Teichmuller, Deutsch. Math. 1 (1936), 362-388.
[Wa71] W. Waterhouse, Automorphism schemes and forms of Witt Lie algebras, J. Algebra 17 (1971), 34–40.
[Wa75] W. Waterhouse, The structure of inseparable field extensions, Trans. Amer. Math. Soc. 211 (1975, 39–56.
[We65] M. Weisfeld, Purely inseparable extensions and higher derivations, Trans. Amer. Math. Soc. 116 (1965), 435–449.
[Yu70] Shuen Yuan, Inseparable Galois theory of exponent one, Trans. Amer. Math. Soc. 149 (1970), 163–170.

Department of Mathematics and Statistics, University at Albany, Albany, NY 12222
E-mail address: lchilds@albany.edu