Applications of the Green’s function in thermoacoustics

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1. Time history calculations
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2. Stability analysis for individual modes
1. Time history calculations
1.1. Green’s function method

acoustic analogy equation for the velocity potential $\Phi$

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = -\frac{\gamma - 1}{c^2} \frac{q'(x,t)}{c^2}$$

forcing term

fluctuations of rate of heat release (per unit mass)

alternative form for the acoustic pressure $p'$

$$\frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = \frac{\gamma - 1}{c^2} \frac{p\partial q'}{c^2 \partial t}(x,t)$$

forcing term

Forced PDEs → suitable for Green’s function approach
Governing equation for the Green’s function

\[
\frac{1}{c^2} \frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial x^2} = \delta(x - x^*)\delta(t - t^*)
\]

Combine with equation for \( \Phi \)

\[
\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} = -\frac{\gamma - 1}{c^2} q'(x,t)
\]

to get integral equation

\[
\phi(x,t) = -\frac{\gamma - 1}{c^2} \int_{t^*=-\infty}^{t} \int_{x^*=-\infty}^{x} G(x,x^*,t-t^*) q'(x^*,t^*) \, dx^* \, dt^*
\]

can be calculated for a compact heat source at \( x_q \),

\[
q'(x,t) = q(t) \delta(x - x_q)
\]
\[
\phi(x,t) = -\frac{\gamma - 1}{c^2} \int_{t^* = 0}^{t} G(x, x_q, t - t^*) q(t^*) \, dt^*
\]

\[
\frac{\partial \phi}{\partial x}\bigg|_{x=x_q}
\]

\[
\frac{\partial G}{\partial x}\bigg|_{x=x_q, x^*=x_q}
\]

equation for velocity at \(x_q\):

\[
u_q(t) = -\frac{\gamma - 1}{c^2} \int_{t^* = 0}^{t} \frac{\partial G}{\partial x}\bigg|_{x=x_q, x^*=x_q} q(t^*) \, dt^*
\]

Assume time-lag law: \(q(t) = q(u_q(t - \tau))\) (linear or nonlinear)

\[
u_q(t) = -\frac{\gamma - 1}{c^2} \int_{t^* = 0}^{t} \frac{\partial G}{\partial x}\bigg|_{x=x_q, x^*=x_q} q(u_q(t - \tau)) \, dt^*
\]

velocity at current time velocity at earlier time

This is an integral equation for \(u_q(t)\) (Volterra equation).
Solve with iteration: time-stepping method

Discretise: \( t \to t_m = 0, \Delta t, 2\Delta t, \ldots \ m\Delta t \)

\( t^* \to t^*_j = 0, \Delta t, 2\Delta t, \ldots \ j\Delta t \)

\[
\int_{t^*=0}^{t} \ldots \ dt^* \to \sum_{j=1}^{m} \ldots \Delta t
\]

Then

\[
u_q(t_m) = \sum_{j=0}^{m} g(t_m - t_j^*) \ q(t_j^*) \Delta t
\]

This can be solved iteratively.

Initial conditions: the initial heat pulse is known, \( q(t)\big|_{t=0} = q_0 \)

the velocity before \( t = 0 \) is zero,

\[
u_q(t - \tau) = 0 \quad \text{for} \quad t - \tau < 0
\]
First few iteration steps:

$m = 0, t_m = 0$ :

$q(0) = q_0$
$u_q(0) = g(0)q(0)$

$m = 1, t_m = \Delta t$ :

$q(\Delta t) = q(u_q(\Delta t - \tau))$
$u_q(\Delta t) = g(\Delta t)q(0) + g(0)q(\Delta t)$

$m = 2, t_m = 2\Delta t$ :

$q(\Delta t) = q(u_q(2\Delta t - \tau))$
$u_q(2\Delta t) = g(2\Delta t)q(0) + g(\Delta t)q(\Delta t) + g(0)q(2\Delta t)$

.....

Problem: As $m$ increases, more and more terms need to be calculated and added.
Idea: Find a more efficient iteration scheme by exploiting the fact that we know the Green's function analytically as a superposition of modes.

\[ G(x, x^*, t - t^*) = \sum_{n=1}^{N} G_n(x, x^*) e^{-i\omega_n(t-t^*)} \text{ for } t > t^* \]

\( \omega_n \): eigenfrequencies

\( G_n \): Green's function amplitudes

\( N \): maximum number of modes considered

Introduce abbreviation  
\[ g_n = \frac{-\gamma - 1}{c^2} \frac{\partial G_n}{\partial x} \bigg|_{x=x_q} \text{, then} \]

\[ u_q(t) = \int_{t^*=0}^{t} \sum_{n=1}^{N} g_n e^{-i\omega_n(t-t^*)}q(t^*)dt^* \]

Collect terms with \( t^* \):
\[ u_q(t) = \sum_{n=1}^{N} g_n e^{-i\omega_n t} \left( \int_{t^*=0}^{t} e^{i\omega t^*} q(t^*) dt^* \right) \]

\[ I_n(t) \] (abbreviation)

\[ \text{split up integration range: } \int_{t^*=0}^{t} = \int_{t^*=0}^{t-\Delta t} + \int_{t^*=t-\Delta t}^{t} \]

Then

\[ I_n(t) = \int_{t^*=0}^{t-\Delta t} e^{i\omega t^*} q(t^*) dt^* + \int_{t^*=t-\Delta t}^{t} e^{i\omega t^*} q(t^*) dt^* \]

\[ = I_n(t-\Delta t) \]

\[ \approx q(t) \int_{t^*=t-\Delta t}^{t} e^{i\omega t^*} dt^* \]

\[ = \frac{e^{i\omega t}}{i\omega_n} \left( 1 - e^{-i\omega_n \Delta t} \right) \]
Iteration scheme

\[ u_q(t) = \sum_{n=1}^{N} g_n e^{-i\omega_n t} I_n(t) \]

with

\[ I_n(t) = I_n(t - \Delta t) + q(t) \frac{e^{i\omega_n t}}{i\omega_n} \left(1 - e^{-i\omega_n \Delta t}\right) \]

\[ q(t) = q(u_q(t - \tau)) \]

Only \( N \) terms need to be calculated in each iteration step.

**Example: Rijke tube**

ideal open ends: \( \phi(0, t) = 0, \phi(L, t) = 0 \)

hot-wire gauze at \( x_q \)

heat release characteristic from hot wire theory:

\[ q(t) = a + b \sqrt{|u| + u_q(t - \tau)} \quad \text{nonlinear!} \]

constants, mean flow velocity
Time history of the velocity

- Exponential increase
- Flow reversal
Time history of the heat release rate (fluctuating part only)

- Heat release rate: \( \text{m}^2 \text{s}^{-2} \)
- Initial heat pulse

Graph showing the fluctuating part of the heat release rate over time.
Summary
If we know the
- tailored Green’s function
- heat release law

of a thermoacoustic system, then we can calculate the
time histories

- \( u_q(t) \) (velocity at the heat source)
- \( q(t) \) (heat release rate)

from a straightforward iteration scheme, stepping forward in
time.
1.2. Comparison with Galerkin method

Idea: expand the acoustic field in terms of idealised eigenmodes

pressure: $p'(x,t) = \sum_{n=1}^{N} p_n(t)\sin(n\pi x)$

velocity: $u'(x,t) = \sum_{n=1}^{N} u_n(t)\cos(n\pi x)$

These are normalised: $\omega_n = n\pi$

practical tubes have: end correction
- temperature gradient
- change in cross-sectional area

... 

Note: The Galerkin modes are an approximation of the real modes in the tube.
substitute into conservation equations

mass: \[ \frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = (\gamma - 1)q(t)\delta(x - x_q) \]

\[ \sum_{n=1}^{N} \left[ \dot{\rho}_n(t) - \gamma M n\pi u_n(t) \right] \sin(n\pi x) = (\gamma - 1)q(t)\delta(x - x_q) \]

orthogonality of \( \sin(n\pi x) \) gives (see aside)
\[ \dot{\rho}_n(t) - \gamma M n\pi u_n(t) = (\gamma - 1)q(t)\sin(n\pi x_q) \]

momentum: \[ \gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial t} = 0 \]

\[ \sum_{n=1}^{N} \left[ \gamma M \dot{u}_n(t) + n\pi p_n(t) \right] \cos(n\pi x) = 0 \]

orthogonality of \( \cos(n\pi x) \) gives
\[ \dot{u}_n(t) + \frac{n\pi}{\gamma M} p_n(t) = 0 \]
Aside

Orthogonality of the eigenfunctions:

\[
\int_{x=0}^{1} \sin(n\pi x)\sin(n'\pi x)\,dx = \begin{cases}
0 & \text{if } n \neq n' \\
\frac{1}{2} & \text{if } n = n'
\end{cases} \quad \text{(same for } \cos(n\pi x))
\]

This will help to separate the modes, e.g. in mass equation:

\[
\sum_{n=1}^{N} \left[ \ddot{p}_n(t) - \gamma M n \pi u_n(t) \right] \sin(n\pi x) = (\gamma - 1) q(t) \delta(x - x_q)
\]

Multiply both sides by \(\sin(n'\pi x)\), integrate over tube \(\int_{x=0}^{1} \cdots \,dx\) = 0, unless \(n = n'\)

\[
\sum_{n=1}^{N} \left[ \ddot{p}_n(t) - \gamma M n \pi u_n(t) \right] \int_{x=0}^{1} \sin(n\pi x)\sin(n'\pi x)\,dx = (\gamma - 1) q(t) \sin(n'\pi x_q)
\]

Only the term \(n = n'\) remains,

\[
\ddot{p}_{n'}(t) - \gamma M n' \pi u_{n'}(t) = (\gamma - 1) q(t) \sin(n'\pi x_q)
\]
From conservation equations:
\[ \dot{u}_n(t) + \frac{n\pi}{\gamma M} p_n(t) = 0 \]
\[ \dot{p}_n(t) - \gamma M n \pi u_n(t) = (\gamma - 1) q(t) \sin(n\pi x_q) \]
in matrix form:
\[
\begin{bmatrix}
\dot{u}_1 \\
\dot{p}_1 \\
\dot{u}_2 \\
\dot{p}_2 \\
\vdots
\end{bmatrix}
= 
\begin{bmatrix}
0 & -\frac{\pi}{\gamma M} & 0 & 0 & \cdots \\
\frac{\pi}{\gamma M} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -\frac{2\pi}{\gamma M} & \cdots \\
0 & 0 & \frac{2\pi}{\gamma M} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
u_1 \\
p_1 \\
u_2 \\
p_2 \\
\vdots
\end{bmatrix}
+ q(t)
\begin{bmatrix}
0 \\
(\gamma - 1) \sin \pi x_q \\
0 \\
(\gamma - 1) \sin \pi x_q \\
\vdots
\end{bmatrix}
\]
or
\[ \dot{\Psi}(t) = \mathbf{M} \Psi(t) + q(t) \mathbf{F} \]
Matrix differential equation for \( u_1(t), u_2(t), \ldots, p_1(t), p_2(t), \ldots \)
Solution

Iteration by time stepping method
discretise: \( t = 0, \Delta t, 2\Delta t, \ldots \)

\[
\Psi(t) \approx \frac{\Psi(t + \Delta t) - \Psi(t)}{\Delta t}
\]

The resulting equation

\[
\Psi(t + \Delta t) = (\mathbf{M} \Delta t + \mathbf{I})\Psi(t) + q(t) \mathbf{F}
\]

with

\[
q(t) = q\left( \sum_{n=1}^{N} u_n(t - \tau) \cos(n\pi x_q) \right)
\]

can be solved iteratively.

Initial conditions:

the initial heat pulse is known, \( q(t)\big|_{t=0} = q_0 \).

the amplitudes \( u_n(0) \) and \( p_n(0) \) are zero for all modes.
Comparison with the Green’s function method
Both methods apply to compact heat sources and lead to explicit iteration schemes stepping forward in time.

Green’s function modes: real physical modes, with
\( \omega_n \): resonance frequencies
\( G_n(x,x^*) \): spatial distribution of modal amplitudes
truncation of sum:
\[
\sum_{n=1}^{\infty} \to \sum_{n=1}^{N}
\]
\( N \): largest relevant mode number, given by observation

Galerkin modes: approximation of physical modes, with
\( \omega_n = n\pi \): resonance frequencies of ideal tube
\( \sin(n\pi x) \): approximate distribution of modal amplitudes
truncation of sum:
\[
\sum_{n=1}^{\infty} \to \sum_{n=1}^{N}
\]
- what is \( N \)?
2. Stability analysis of individual modes

Integral equation for the velocity

\[ u_q(t) = \text{Re} \left( \int_{t^*=0}^{t} \sum_{n=1}^{N} g_n e^{-i\omega_n(t-t^*)}q(t^*)dt^* \right) \]

omitted in earlier material

\[ \text{Re} [...] = \frac{1}{2} \left( [...] + \overline{[...]} \right) \] complex conjugate

Assume that only mode \( n \) is present:

\[ u_n(t) = \frac{1}{2} \int_{t^*=0}^{t} g_n e^{-i\omega_n(t-t^*)}q(t^*)dt^* + \frac{1}{2} \int_{t^*=0}^{t} g_n e^{i\omega_n(t-t^*)}q(t^*)dt^* \]

or

\[ u_n(t) = \frac{1}{2} \left[ I_n(t) + \overline{I_n(t)} \right] \] with \( I_n(t) = \int_{t^*=0}^{t} g_n e^{-i\omega_n(t-t^*)}q(t^*)dt^* \)
Conversion of integral equation into differential equation

**Step 1:** Determine $\frac{\partial I_n}{\partial t}$, noting that $t$ appears in the integrand and in the integration limit. $\frac{\partial I_n}{\partial t} = -i\omega_n I_n + g_n q(t)$

**Step 2:** Use this result to calculate $\dot{u}_n$ and $\ddot{u}_n$.

\[
\dot{u}_n = \frac{1}{2} \left[ (g_n + \overline{g_n})q(t) - i\omega_n I_n + i\overline{\omega_n I_n} \right]
\]

\[
\ddot{u}_n = \frac{1}{2} \left[ (g_n + \overline{g_n})\ddot{q}(t) - (i\omega_n g_n + i\overline{\omega_n g_n})q(t) - \omega_n^2 I_n - \overline{\omega_n^2 I_n} \right]
\]

**Step 3:** Multiply as indicated and add the resulting equations.

\[
u_n(t) = \frac{1}{2} \left[ I_n(t) + \overline{I_n(t)} \right] \cdot (\omega_n \overline{\omega_n})
\]

\[
\dot{u}_n = \frac{1}{2} \left[ (g_n + \overline{g_n})q(t) - i\omega_n I_n + i\overline{\omega_n I_n} \right] \cdot (i\omega_n)
\]

This eliminates $\overline{I_n}$ and gives an expression for $I_n$,

\[
I_n = \frac{1}{\omega_n + \overline{\omega_n}} \left[ 2(iu_n + \overline{\omega_n u_n}) - i(g_n + \overline{g_n})q(t) \right]
\]

**Step 4:** In the equation for $\ddot{u}_n$, substitute for $I_n$, and simplify.
Result

\[ \ddot{u}_n - 2\text{Im}(\omega_n) \dot{u}_n + |\omega_n|^2 u_n = -\text{Im}(\omega_n \overline{g_n}) q(t) + \text{Re}(g_n) \dot{q}(t) \]

damped harmonic oscillator \hspace{1cm} forcing term

Assume heat release law \[ q(t) = \left[ n_1 u(t - \tau) - n_0 u(t) \right] \]

amplitude-dependent coefficients can be obtained e.g. from FDF measurements

substitute into oscillator equation:

\[ \ddot{u}_n + \left[ -2\text{Im}(\omega_n) + n_0 \text{Re}(g_n) \right] \dot{u}_n + \left[ |\omega_n|^2 - n_0 \text{Im}(\omega_n \overline{g_n}) \right] u_n = \]

\[ = c_1 \]

\[ = c_0 \]

\[ = b_0 \]

\[ = b_1 \]
\[ \ddot{u}_n(t) + c_1 \dot{u}_n(t) + c_0 u_n(t) = b_0 u_n(t - \tau) + b_1 \dot{u}_n(t - \tau) \]

We look for steady limit cycle solutions: \( u_n(t) = A \cos(\Omega t) \approx \omega_n \)

For any time-lag:
\[ u_n(t - \tau) = A \cos(\Omega(t - \tau)) = (\cos(\Omega \tau)) u_n(t) - \frac{\sin(\Omega \tau)}{\Omega} \dot{u}_n(t) \]
\[ \dot{u}_n(t - \tau) = -\Omega A \sin(\Omega(t - \tau)) = (\Omega \sin(\Omega \tau)) u_n(t) + (\cos(\Omega \tau))\dot{u}_n(t) \]

ODE for \( u_n(t) \):
\[ \ddot{u}_n(t) + [c_1 + b_0 \frac{\sin(\Omega \tau)}{\Omega} - b_1 \cos(\Omega \tau)] \dot{u}_n(t) + [c_0 - b_0 \cos(\Omega \tau) - b_1 \Omega \sin(\Omega \tau)] u_n(t) = 0 \]

\( a_0 \): oscillation frequency (squared)
\( a_1 \): damping coefficient, amplitude-dependent

\( a_1 > 0 \): stability, \( a_1 < 0 \): instability
The stability behaviour can be examined at different amplitudes for various system parameters.

**Example:** ¼ wave resonator with variable length and amplitude-dependent time-lag law

\[ \tau = \tau_0 + \tau_1 A^2, \quad A: \text{velocity amplitude} \]
Summary

Analysis works well for weakly coupled modes.

Straightforward stability predictions.

Suitable for cases where the nonlinear heat release law is given in terms of amplitude-dependent coefficients.
Thank you!

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