Network Models in Thermoacoustics

Ph.D. Camilo Silva
Prof. Wolfgang Polifke
Why changing an injector position could make a flame unstable?
Can entropy couple with the flame through acoustic waves?

How to study combustion instabilities?
How to study combustion instabilities?

Experiments

High fidelity CFD

Network models

All together may be the best solution!!

For example …

Helicopter Engine
How to study combustion instabilities?

- Experiments
- High fidelity CFD
- Network models

All together may be the best solution!!

For example …

Helicopter Engine

Network Models

Flame dynamics from experiments or CFD

or Network Models
What we want to study?

✓ Thermoacoustics

Of longitudinal, transversal, azimuthal or radial acoustic waves?
What we want to study?

✓ Thermoacoustics

✓ Of longitudinal plane acoustic waves

Of short or long wavelengths?
What we want to study?

✓ Thermoacoustics

✓ Of longitudinal plane acoustic waves

✓ Of long wavelengths

Main Assumption

Acoustic compactness in most elements of the thermoacoustic system
Thermoacoustic Network models

Decompose the System

Understand the System
<table>
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<tr>
<th>Full Thermoacoustic System</th>
<th>Understand the System</th>
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<td>Decompose the System</td>
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- White Box
- Black Box
- Gray Box
- Ducts
- Compact Flames
- Nozzles
- Joints
- ...
**Full Thermoacoustic System**

| Understand the System |

**Thermoacoustic Network models**

| Decompose the System |

**Acoustic two-port element**

| Ducts |
| Compact Flames |
| Nozzles |
| Joints |

**WHITE BOX**

\[
\begin{align*}
\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) &= 0 \\
\frac{\partial}{\partial x} (\rho u A) + \frac{\partial}{\partial x} (\rho u A^2) &= -A \frac{\partial p}{\partial x} \\
\frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) &= \frac{A}{T} \dot{q} \\
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) &= A \dot{q} - A \frac{\partial p}{\partial t}
\end{align*}
\]

**Model Acoustic and entropy waves**

**Quasi 1D Conservation Equations**
Gather all elements in a single matrix and compute acoustic response of the ensemble.

Note that under a suitable treatment, tens of elements can reduce to a 4 x 4 matrix!
**Full Thermoacoustic System**

Understand the System

**Thermoacoustic Network models**

Decompose the System

**Acoustic two-port element**

White Box

Black Box

Gray Box

Ducts

Compact Flames

Nozzles

Joints

…

**WHITE BOX**

\( \dot{\theta} ' \rho ' u ' p ' s ' \)

Model Acoustic and entropy waves

**Quasi 1D Conservation Equations**

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\begin{align*}
\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) &= 0 \\
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\end{align*}
\]

**CONNEXIONS**

\[
\begin{bmatrix}
1 & -R_{in} & 0 & 0 \\
0 & 0 & -R_{out} & 1 \\
T_{11} & T_{12} & -1 & 0 \\
T_{21} & T_{22} & 0 & -1
\end{bmatrix}
\begin{bmatrix}
f_0 \\
g_0 \\
f_3 \\
g_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Gather all elements in a single matrix and compute acoustic response of the ensemble.

**STABILITY ANALYSIS.**

Study stability of the system
OUTLINE

WHITE BOX

\( \dot{Q}' \rho' u' p' s' \)

Model Acoustic
and entropy waves

\[
\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) = 0 \\
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\frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) = A \dot{q} - A \frac{\partial p}{\partial t} \\
\frac{\partial}{\partial t} (\rho h_1 A) + \frac{\partial}{\partial x} (\rho u h_1 A) = A \dot{q} - A \frac{\partial p}{\partial t}
\]

Quasi 1D Conservation Equations
Spatial integration and the compact assumption
Linearization
Further Assumptions

Definition of waves
Isentropic ducts
Boundary conditions
Modeling of Flame dynamics

CONNEXIONS

Gather all elements in a single matrix and compute acoustic response of the ensemble.

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\begin{pmatrix}
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\end{pmatrix}
\]

STABILITY ANALYSIS.

Study stability of the system

Freq. Unstable Region

Neg. Pos.

Growth rate
**WHITE BOX**

\( \dot{Q}' \ \rho' \ u' \ p' \ s' \)

Model Acoustic and entropy waves

---

### Quasi 1D Conservation Equations

Spatial integration and the compact assumption

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\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) &= A \dot{q} - A \frac{\partial p}{\partial t}
\end{align*}
\]

\[
f = B^+ e^{-i \omega x / (1 + M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho c} + \dot{\mu} \right)
\]

\[
g = B^- e^{i \omega x / (1 - M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho c} - \dot{\mu} \right)
\]

\[
\dot{Q} = \bar{\rho} \bar{u} A_1 c_p \bar{T} \left( \frac{T_2}{T_1} - 1 \right) = \bar{u} A_1 \frac{\gamma p_1}{(\gamma - 1)} \left( \frac{T_2}{T_1} - 1 \right)
\]

---

### CONNEXIONS

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\begin{bmatrix}
1 & -R_{\text{in}} & 0 & 0 \\
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Gather all elements in a single matrix and compute acoustic response of the ensemble.

---

### STABILITY ANALYSIS.

Study stability of the system

\[
\begin{array}{c}
\text{Freq.} \\
\text{Unstable Region}
\end{array}
\]

\[
\begin{array}{c}
\text{Neg.} \\
\text{Pos.}
\end{array}
\]

\[
\text{Growth rate}
\]

---
Quasi 1D Conservation Equations

From full Navier-Stokes equations

**Assumptions**
- No viscous terms
- Quasi-1D

Mass

\[
\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) = 0
\]
Quasi 1D Conservation Equations

From full Navier-Stokes equations

Assumptions
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Mass
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\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) = 0
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Momentum
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\frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) = -A \frac{\partial p}{\partial x}
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From full Navier-Stokes equations

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**Momentum**
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\]

**Entropy**
\[
\frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) = \frac{A}{T} \dot{q}
\]
Quasi 1D Conservation Equations

From full Navier-Stokes equations

**Assumptions**
- No viscous terms
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Mass

$$\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) = 0$$

Momentum

$$\frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) = -A \frac{\partial p}{\partial x}$$

Entropy

$$\frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) = \frac{A}{T} \dot{q}$$

Total Enthalpy

$$\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t}$$
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**Total Enthalpy**

**Assumptions**
- No viscous terms
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Quasi 1D Conservation Equations

Total Enthalpy

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\]

Assumptions
- No viscous terms
- Quasi-1D

Mass

\[
\frac{1}{\rho} \frac{D \rho}{Dt} = - \frac{1}{A} \frac{\partial}{\partial x} (u A)
\]

Momentum

\[
\frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) = -A \frac{\partial p}{\partial x}
\]

Entropy

\[
\frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) = \frac{A}{T} \dot{q}
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Quasi 1D Conservation Equations

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Total Enthalpy

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Mass

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\frac{1}{\rho} \frac{D \rho}{Dt} = - \frac{1}{A} \frac{\partial}{\partial x} (u A)
\]

Momentum

\[
\frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) = -A \frac{\partial p}{\partial x}
\]

Entropy

\[
\frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) = A \frac{\dot{q}}{T}
\]
Quasi 1D Conservation Equations

Mass
\[ \frac{1}{\rho} \frac{D \rho}{Dt} = -\frac{1}{A} \frac{\partial}{\partial x} (uA) \]

Momentum
\[ \rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} \]

Entropy
\[ \rho T \frac{Ds}{Dt} = \dot{q} \]

Assumptions
- No viscous terms
- Quasi-1D
Quasi 1D Conservation Equations

Total Enthalpy

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\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t}
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Momentum

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Total Enthalpy

\[ \frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t} \]

Momentum

\[ \rho \frac{D u}{D t} = - \frac{\partial p}{\partial x} \]

Mass

\[ \frac{1}{\rho} \frac{D \rho}{D t} = - \frac{1}{A} \frac{\partial}{\partial x} (uA) \]

Entropy

\[ \rho T \frac{D s}{D t} = \dot{q} \]
Quasi 1D Conservation Equations

Total Enthalpy

$$\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t}$$

Momentum

$$\frac{D u}{D t} = -\frac{\partial p}{\partial x}$$

Mass

$$\frac{1}{\rho} \frac{D \rho}{D t} = -\frac{1}{A} \frac{\partial}{\partial x} (uA)$$

Entropy

$$\rho T \frac{D s}{D t} = \dot{q}$$

$$\frac{1}{c_p} \frac{D s}{D t} = \frac{1}{\gamma p} \frac{D p}{D t} - \frac{1}{\rho} \frac{D \rho}{D t}$$

2nd law thermodynamics

Assumptions
- No viscous terms
- Quasi-1D
Quasi 1D Conservation Equations

**Total Enthalpy**
\[
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t}
\]

**Momentum**
\[
\frac{Du}{Dt} = -\frac{\partial p}{\partial x}
\]

**Mass**
\[
\frac{1}{\rho} \frac{D\rho}{Dt} = -\frac{1}{A} \frac{\partial}{\partial x} (uA)
\]

**Entropy**
\[
\rho T \frac{Ds}{Dt} = \dot{q}
\]
\[
\frac{Ds}{cp} \frac{Dt}{Dt} = \frac{1}{\gamma p} \frac{Dp}{Dt} - \frac{1}{\rho} \frac{D\rho}{Dt}
\]

**Assumptions**
- No viscous terms
- Quasi-1D
Quasi 1D Conservation Equations

**Total Enthalpy**
\[
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A\dot{q} - A \frac{\partial p}{\partial t}
\]

**Momentum**
\[
\rho \frac{Du}{Dt} = - \frac{\partial p}{\partial x}
\]

**Entropy**
\[
\frac{A}{\gamma p} \frac{Dp}{Dt} + \frac{\partial}{\partial x} (uA) = \frac{(\gamma - 1)}{\gamma p} \dot{q} A
\]

**Assumptions**
- No viscous terms
- Quasi-1D

**Thermodynamics**
- Total Enthalpy
- Mass
- 2nd law therm.
Quasi 1D Conservation Equations

**Assumptions**
- No viscous terms
- Quasi-1D

**Total Enthalpy**
\[
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t}
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**Momentum**
\[
\frac{Du}{Dt} = - \frac{\partial p}{\partial x}
\]

**Entropy**
\[
\frac{A}{\gamma p} \frac{Dp}{Dt} + \frac{\partial}{\partial x} (u A) = \frac{(\gamma - 1)}{\gamma p} \dot{q} A
\]

Not convenient … we have to reorganize somehow
Quasi 1D Conservation Equations

Total Enthalpy

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\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t}
\]

Momentum

\[
\frac{\rho D u}{D t} = - \frac{\partial p}{\partial x}
\]

Entropy

\[
\frac{A}{\gamma p} \frac{D p}{D t} + \frac{\partial}{\partial x} (u A) = \frac{(\gamma - 1)}{\gamma p} \dot{q} A
\]

Mass

\[
\frac{A}{\gamma p} \frac{\partial p}{\partial t} + \frac{A u}{\gamma p} \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} (u A) = \frac{(\gamma - 1)}{\gamma p} \dot{q} A
\]

2nd law therm.

\[
A \frac{\partial \ln (p^{1/\gamma})}{\partial t} + \frac{1}{p^{1/\gamma}} \frac{\partial}{\partial x} \left( p^{1/\gamma} u A \right) = \frac{(\gamma - 1)}{\gamma p} \dot{q} A
\]

Assumptions

- No viscous terms
- Quasi-1D
Quasi 1D Conservation Equations

Assumptions
- No viscous terms
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### Total Enthalpy
\[
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A\dot{q} - A\frac{\partial p}{\partial t}
\]

### Momentum
\[
\frac{D u}{D t} = -\frac{\partial p}{\partial x}
\]

### Entropy
### Mass
### 2nd law therm.
OUTLINE

WHITE BOX

\[ \dot{Q}' \rho' u' p' s' \]

Model Acoustic and entropy waves

\[
\begin{align*}
\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) &= 0 \\
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\]

\[ f = B^+ e^{-i \omega x / c (1 + M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho c} + \dot{u} \right) \]

\[ g = B^- e^{i \omega x / c (1 - M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho c} - \dot{u} \right) \]

\[ \ddot{Q} = \bar{\rho} \bar{u}_1 A_1 c_p T_1 \left( \frac{T_2}{T_1} - 1 \right) = \bar{u}_1 A_1 \frac{\gamma \bar{p}_1}{(\gamma - 1)} \left( \frac{T_2}{T_1} - 1 \right) \]

\[ \frac{\dot{Q}}{\dot{Q}} = \bar{\rho} \bar{u}_1 A_1 c_p T_1 \left( \frac{T_2}{T_1} - 1 \right) = \bar{u}_1 A_1 \frac{\gamma \bar{p}_1}{(\gamma - 1)} \left( \frac{T_2}{T_1} - 1 \right) \]

Quasi 1D Conservation Equations

Spatial integration and the compact assumption

Linearization

Further Assumptions

Definition of waves

Isentropic ducts

Boundary conditions

Modeling of Flame dynamics

CONNEXIONS

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\begin{bmatrix}
1 & -R_{in} & 0 & 0 \\
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\]

Gather all elements in a single matrix and compute acoustic response of the ensemble.

STABILITY ANALYSIS.

Study stability of the system
Spatial integration and the compact assumption

Total Enthalpy

\[
\frac{\partial}{\partial t}(\rho h_t A) + \frac{\partial}{\partial x}(\rho u h_t A) = Aq - A\frac{\partial p}{\partial t}
\]

Assumptions

- No viscous terms
- Quasi-1D

\[
\int_{x_1}^{x_2} \frac{\partial}{\partial t}(\rho h_t A) \, dx + \int_{x_1}^{x_2} \frac{\partial}{\partial x}(\rho u h_t A) \, dx = \int_{x_1}^{x_2} Aq \, dx - \int_{x_1}^{x_2} A\frac{\partial p}{\partial t} \, dx
\]
Spatial integration and the compact assumption

Total Enthalpy
\[
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t}
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\[
\int_{x_1}^{x_2} \frac{\partial}{\partial t} (\rho h_t A) \ dx + \int_{x_1}^{x_2} \frac{\partial}{\partial x} (\rho u h_t A) \ dx = \int_{x_1}^{x_2} A \dot{q} \ dx - \int_{x_1}^{x_2} A \frac{\partial p}{\partial t} \ dx
\]

Compact element if
\[
(x_1 - x_2) = \Delta x \ll \lambda
\]

Assumptions
- No viscous terms
- Quasi-1D
Spatial integration and the compact assumption

Total Enthalpy

\[
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t}
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Compact element if

\[
(x_1 - x_2) = \Delta x \ll \lambda
\]

Compact assumption means to neglect those integral terms for which

Its inside quantity is bounded

Assumptions

- No viscous terms
- Quasi-1D
Spatial integration and the compact assumption

Total Enthalpy

\[ \frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = \dot{A} q - A \frac{\partial p}{\partial t} \]

Assumptions
- No viscous terms
- Quasi-1D

Compact element if

\[ (x_1 - x_2) = \Delta x \ll \lambda \]

Compact assumption means to neglect those integral terms for which

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Spatial integration and the compact assumption

Total Enthalpy

\[
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t}
\]

\[
\int_{x_1}^{x_2} \frac{\partial}{\partial t} (\rho h_t A) \, dx + \int_{x_1}^{x_2} \frac{\partial}{\partial x} (\rho u h_t A) \, dx = \int_{x_1}^{x_2} A \dot{q} \, dx - \int_{x_1}^{x_2} A \frac{\partial p}{\partial t} \, dx
\]

\[
[\rho u h_t A]_{x_1}^{x_2} = \dot{Q}
\]

where \( \dot{Q} = \int_{x_1}^{x_2} \dot{q} A \, dx \)
Spatial integration and the compact assumption

Total Enthalpy

\[ [\rho u_h A]_{x_1}^{x_2} = \dot{Q} \]

Entropy

\[ A p^{1/\gamma} \frac{\partial \ln(p^{1/\gamma})}{\partial t} + \frac{\partial}{\partial x} \left( p^{1/\gamma} u A \right) = \frac{(\gamma - 1)}{\gamma \rho} \dot{q} A p^{1/\gamma} \]

Mass

2nd law therm.

Assumptions
- No viscous terms
- Quasi-1D
- Compactness
Spatial integration and the compact assumption

**Total Enthalpy**

\[ [\rho u h_t A]_{x_1}^{x_2} = \dot{Q} \]

**Entropy**

\[ A p^{1/\gamma} \frac{\partial \ln(p^{1/\gamma})}{\partial t} + \frac{\partial}{\partial x} \left( p^{1/\gamma} u A \right) = \frac{(\gamma - 1)}{\gamma p} \dot{q} A p^{1/\gamma} \]

**Assumptions**
- No viscous terms
- Quasi-1D
- Compactness

**2nd law therm.**

\[ \int_{x_1}^{x_2} A p^{1/\gamma} \frac{\partial \ln(p^{1/\gamma})}{\partial t} \, dx + \int_{x_1}^{x_2} \frac{\partial}{\partial x} \left( p^{1/\gamma} u A \right) \, dx = \int_{x_1}^{x_2} \frac{(\gamma - 1)}{\gamma p} \dot{q} A p^{1/\gamma} \, dx \]
Spatial integration and the compact assumption

Total Enthalpy

\[ [\rho u h_t A]^x_2 = \dot{Q} \]

Assumptions

- No viscous terms
- Quasi-1D
- Compactness

Entropy

Mass

2nd law therm.

\[ A p^{1/\gamma} \frac{\partial \ln(p^{1/\gamma})}{\partial t} + \frac{\partial}{\partial x} \left( p^{1/\gamma} u A \right) = \frac{(\gamma - 1)}{\gamma p} \dot{q} A p^{1/\gamma} \]

\[ \int_{x_1}^{x_2} A p^{1/\gamma} \frac{\partial \ln(p^{1/\gamma})}{\partial t} dx + \int_{x_1}^{x_2} \frac{\partial}{\partial x} \left( p^{1/\gamma} u A \right) dx = \int_{x_1}^{x_2} \frac{(\gamma - 1)}{\gamma p} \dot{q} A p^{1/\gamma} dx \]

\[ [p^{1/\gamma} u A]^x_2 = \int_{x_1}^{x_2} \frac{(\gamma - 1)}{\gamma p} \dot{q} A p^{1/\gamma} dx \]
Spatial integration and the compact assumption

Assumptions

- No viscous terms
- Quasi-1D
- Compactness

Total Enthalpy

\[ [\rho u h_t A]_{x_1}^{x_2} = \dot{Q} \]

Entropy

\[ [p^{1/\gamma} u A]_{x_1}^{x_2} = \int_{x_1}^{x_2} \frac{(\gamma - 1)}{\gamma p} \dot{q} A p^{1/\gamma} \, dx \]

Mass

2nd law therm.
OUTLINE

WHITE BOX
\[ \dot{Q}' \ p' \ u' \ p' \ s' \]

Model Acoustic and entropy waves

\[ \frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho u A) = 0 \]
\[ \frac{\partial}{\partial t}(\rho u A) + \frac{\partial}{\partial x}(\rho u^2 A) = -A \frac{\partial p}{\partial x} \]
\[ \frac{\partial}{\partial t}(\rho s A) + \frac{\partial}{\partial x}(\rho u s A) = A \frac{\partial \bar{q}}{\partial t} \]
\[ \frac{\partial}{\partial t}(\rho h_t A) + \frac{\partial}{\partial x}(\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t} \]

\[ f = B e^{-i\omega x/\bar{c}(1+M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho \bar{c}} + \dot{u} \right) \]
\[ g = B e^{i\omega x/\bar{c}(1-M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho \bar{c}} - \dot{u} \right) \]
\[ \dot{Q} = \bar{\rho}_1 \bar{u}_1 A_1 c_p \bar{T}_1 \left( \frac{\bar{T}_2}{\bar{T}_1} - 1 \right) = \bar{u}_1 A_1 \frac{\gamma \bar{p}_1}{(\gamma - 1)} \left( \frac{\bar{T}_2}{\bar{T}_1} - 1 \right) \]

\[ \begin{bmatrix} 1 & -R_{in} & 0 & 0 \\ 0 & 0 & -R_{out} & 1 \\ T_{11} & T_{12} & -1 & 0 \\ T_{21} & T_{22} & 0 & -1 \end{bmatrix} M \begin{bmatrix} f_0 \\ g_0 \\ f_3 \\ g_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

Gather all elements in a single matrix and compute acoustic response of the ensemble.

STABILITY ANALYSIS.

Study stability of the system

\[ \begin{array}{c}
\text{Freq.} \\
\text{Neg.} \\
\text{Pos.} \\
\text{Growth rate} \\
\text{Unstable Region} \\
\end{array} \]
Linearization of Equations

\[ \| \| = \| \| + \| \|' + \mathcal{O}^2 \]
Linearization of Equations

Total Enthalpy

\[ [\rho u h_t A]^x = \dot{Q} \]

Assumptions

- No viscous terms
- Quasi-1D
- Compactness
- Linear acoustics
Linearization of Equations

Total Enthalpy

\[ [\rho u h_t A]_{x_1}^{x_2} = \dot{Q} \]

\[
\dot{m} \left. h'_t \right|_{x_1}^{x_2} = \dot{Q}' \quad \text{or} \quad \frac{\rho'_1}{\bar{\rho}_1} + \frac{u'_1}{\bar{u}_1} + \frac{T'_t}{\bar{T}_{t2}} - \frac{T'_t}{\bar{T}_{t1}} = \frac{\dot{Q}'}{\dot{Q}}
\]

Assumptions

- No viscous terms
- Quasi-1D
- Compactness
- Linear acoustics
**Linearization of Equations**

**Assumptions**

- No viscous terms
- Quasi-1D
- Compactness
- Linear acoustics

Total Enthalpy

$$\left[ \rho u h_t A \right]_{x_1}^{x_2} = \dot{Q}$$

$$\dot{m} \ h'_t |_{x_1}^{x_2} = \dot{Q}' \text{ or } \frac{\rho'_1}{\rho_1} + \frac{u'_1}{u_1} + \frac{T'_t - T'_t}{T_{t2} - T_{t1}} = \frac{\dot{Q}'}{\dot{Q}}$$

$$\frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \tilde{M}_2 \frac{u'_2}{c_2} + (\gamma - 1) \frac{p'_2}{\gamma \bar{p}_2} + \frac{s'_2}{c_p} \right) = \frac{\dot{Q}'}{\dot{Q}} + \frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \tilde{M}_1 \frac{u'_1}{c_1} + (\gamma - 1) \frac{p'_1}{\gamma \bar{p}_1} + \frac{s'_1}{c_p} \right) - \frac{p'_1}{\gamma \bar{p}_1} - \frac{u'_1}{u_1} + \frac{s_1}{c_p}$$

where

$$\chi = \left( 1 + \frac{\gamma - 1}{2} \tilde{M}^2 \right) \quad \text{and} \quad \lambda = \frac{T_2}{T_1}$$
Linearization of Equations

**Total Enthalpy**

\[
\frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_2 \frac{u_2'}{c_2} + (\gamma - 1) \frac{p_2'}{\gamma \bar{p}_2} + \frac{s_2'}{c_p} \right) = \frac{\dot{Q}'}{Q} + \frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_1 \frac{u_1'}{c_1} + (\gamma - 1) \frac{p_1'}{\gamma \bar{p}_1} + \frac{s_1'}{c_p} \right) - \frac{p_1'}{\gamma \bar{p}_1} - \frac{u_1'}{\bar{u}_1} + \frac{s_1}{c_p}
\]

**Entropy**

\[
[p^{1/\gamma} u A]^{x_2}_{x_1} = \int_{x_1}^{x_2} \frac{(\gamma - 1)}{\gamma \bar{p}} \dot{q} A p^{1/\gamma} \ dx
\]

**Mass**

2nd law therm.

**Assumptions**
- No viscous terms
- Quasi-1D
- Compactness
- Linear acoustics
Linearization of Equations

Total Enthalpy

\[
\frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \tilde{M}_2 \frac{u_2'}{c_2} + (\gamma - 1) \frac{p_2'}{\gamma \bar{p}_2} + \frac{s_2'}{c_p} \right) = \frac{\dot{Q}'}{Q} + \\
\frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \tilde{M}_1 \frac{u_1'}{c_1} + (\gamma - 1) \frac{p_1'}{\gamma \bar{p}_1} + \frac{s_1'}{c_p} \right) - \frac{p_1'}{\gamma \bar{p}_1} - \frac{u_1'}{\bar{u}_1} + \frac{s_1}{c_p}
\]

Entropy

\[
\left[ p^{1/\gamma} u A \right]_{x_1}^{x_2} = \int_{x_1}^{x_2} \frac{\gamma - 1}{\gamma p} \dot{q} A p^{1/\gamma} \, dx
\]

Mass

2nd law therm.

A lot of mathematical treats implemented so that after linearizing we get …

\[
A_2 \pi^{-\beta} \left( \frac{\tilde{M}_2 \bar{c}_2 p_2'}{\gamma} + \bar{p}_2 u_2' \right) = A_1 \left( \frac{\tilde{M}_1 \bar{c}_1 p_1'}{\gamma} + \bar{p}_1 u_1' \right) + \beta \dot{Q}' \left( 1 + \frac{(1 - \pi^\beta)}{2\pi^\beta} \right) \\
- \frac{\beta^2}{\bar{p}_1} \tilde{Q} \left[ 1 + \frac{1}{(\alpha + 1)} \left( p_2' \pi^{-(\beta + 1)} - p_1' \right) \right]
\]

where

\[
\pi = \frac{\bar{p}_2}{\bar{p}_1}
\]
Linearization of Equations

Now we can start doing some simplifications

\[
\frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1)\bar{M} \frac{u_2'}{c_2} + (\gamma - 1)\frac{p_2'}{\gamma \bar{p}_2} + \frac{s_2'}{c_p} \right) = \frac{\dot{Q}'}{\bar{Q}} + \\
\frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1)\bar{M} \frac{u_1'}{c_1} + (\gamma - 1)\frac{p_1'}{\gamma \bar{p}_1} + \frac{s_1'}{c_p} \right) - \frac{p_1'}{\gamma \bar{p}_1} - \frac{u_1'}{\bar{u}_1} + \frac{s_1}{c_p}
\]

Assumptions
- No viscous terms
- Quasi-1D
- Compactness
- Linear acoustics

\[
A_2 \pi^{-\beta} \left( \frac{M_2 \bar{c}_2 p_2'}{\gamma} + \bar{p}_2 u_2' \right) = A_1 \left( \frac{M_1 \bar{c}_1 p_1'}{\gamma} + \bar{p}_1 u_1' \right) + \beta \dot{Q}' \left( 1 + \frac{1 - \pi^{\beta}}{2 \pi^{\beta}} \right) \\
- \frac{\beta^2}{\bar{p}_1} \dot{Q}' \left[ 1 + \frac{1}{\alpha + 1} \left( p_2' \pi^{-(\beta + 1)} - p_1' \right) \right]
\]
OUTLINE

WHITE BOX

\[ \dot{Q}' \quad \rho' \quad u' \quad p' \quad s' \]
Model Acoustic and entropy waves

\[ \frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) = 0 \]
\[ \frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) = -A \frac{\partial p}{\partial x} \]
\[ \frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (p u s A) = A \frac{T}{t} \dot{q} \]
\[ \frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (p u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t} \]

\[ f = B^+ e^{-i\omega x/c(1+M)} = \frac{1}{2} \left( \frac{\dot{p}}{\dot{p}c} + \ddot{u} \right) \]
\[ g = B^- e^{i\omega x/c(1-M)} = \frac{1}{2} \left( \frac{\dot{p}}{\dot{p}c} - \ddot{u} \right) \]
\[ \dot{Q} = \bar{\rho}_1 \bar{u}_1 A_1 \bar{c}_p T_1 \left( \frac{T_2}{T_1} - 1 \right) = \bar{u}_1 A_1 \frac{\gamma \bar{p}_1}{(\gamma - 1)} \left( \frac{T_2}{T_1} - 1 \right) \]

Quasi 1D Conservation Equations
Spatial integration and the compact assumption
Linearization

Further Assumptions

Definition of waves
Isentropic ducts
Boundary conditions
Modeling of Flame dynamics

CONNEXIONS

\[
\begin{pmatrix}
1 & -R_{in} & 0 & 0 \\
0 & 0 & -R_{out} & 1 \\
T_{11} & T_{12} & -1 & 0 \\
T_{21} & T_{22} & 0 & -1 \\
\end{pmatrix} M
\]

\[
\begin{pmatrix}
f_0 \\
g_0 \\
f_3 \\
g_3 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]
Gather all elements in a single matrix and compute acoustic response of the ensemble.

STABILITY ANALYSIS.

Study stability of the system

Freq.
Neg.
Pos.
Growth rate
Unstable Region

51
Assumptions

- No viscous terms
- Quasi-1D
- Compactness
- Linear acoustics

\[
\frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_2 \frac{u_2'}{c_2} + (\gamma - 1) \frac{p_2'}{\gamma \bar{p}_2} + \frac{s_2'}{c_p} \right) = \frac{\dot{Q}'}{\bar{Q}} + \\
\frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_1 \frac{u_1'}{c_1} + (\gamma - 1) \frac{p_1'}{\gamma \bar{p}_1} + \frac{s_1'}{c_p} \right) - \frac{p_1'}{\gamma \bar{p}_1} - \frac{u_1'}{\bar{u}_1} + \frac{s_1}{c_p}
\]

Further assumptions

- Isentropic flow

Total Enthalpy

Entropy

Mass

2nd law therm.
Assumptions

- No viscous terms
- Quasi-1D
- Compactness
- Linear acoustics

Total Enthalpy

\[
\frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_2 \frac{u'_2}{c_2} + (\gamma - 1) \frac{p'_2}{\gamma \bar{p}_2} + \frac{s'_2}{c_p} \right) = \frac{\dot{Q}'}{\bar{Q}} + \\
\frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_1 \frac{u'_1}{c_1} + (\gamma - 1) \frac{p'_1}{\gamma \bar{p}_1} + \frac{s'_1}{c_p} \right) - \frac{p'_1}{\gamma \bar{p}_1} - \frac{u'_1}{\bar{u}_1} + \frac{s_1}{c_p}
\]

Entropy

Mass

2nd law therm.

Further assumptions

- Isentropic flow

Total Enthalpy

\[
\frac{1}{1 + \frac{\gamma - 1}{2} \bar{M}_1^2} \left( \bar{M}_1 \frac{u'_1}{c_1} + \frac{p'_1}{\gamma \bar{p}_1} + \frac{s'_1}{c_p} \right) = \frac{1}{1 + \frac{\gamma - 1}{2} \bar{M}_2^2} \left( \bar{M}_2 \frac{u'_2}{c_2} + \frac{p'_2}{\gamma \bar{p}_2} + \frac{s'_2}{c_p} \right)
\]

Entropy

Mass

2nd law therm.

\[
A_1 \left( \bar{\rho}_1 u'_1 + \bar{M}_1 \frac{p'_1}{c_1} \right) = A_2 \left( \bar{\rho}_2 u'_2 + \bar{M}_2 \frac{p'_2}{c_2} \right)
\]
Assumptions

- No viscous terms
- Quasi-1D
- Compactness
- Linear acoustics

Total Enthalpy
\[ \frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_2 \frac{u_2'}{\bar{c}_2} + (\gamma - 1) \frac{p_2'}{\gamma \bar{p}_2} + \frac{s_2'}{c_p} \right) = \frac{\dot{Q}'}{\bar{Q}} + \]

Entropy
\[ \frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_1 \frac{u_1'}{\bar{c}_1} + (\gamma - 1) \frac{p_1'}{\gamma \bar{p}_1} + \frac{s_1'}{c_p} \right) - \frac{p_1'}{\gamma \bar{p}_1} - \frac{u_1'}{\bar{u}_1} + \frac{s_1}{c_p} \]

Mass
\[ A_2 \pi^{-\beta} \left( \frac{M_2 \bar{c}_2 p_2'}{\gamma} + \bar{p}_2 u_2' \right) = A_1 \left( \frac{M_1 \bar{c}_1 p_1'}{\gamma} + \bar{p}_1 u_1' \right) + \beta \dot{Q}' \left( 1 + \frac{(1 - \pi^\beta)}{2 \pi^\beta} \right) - \frac{\beta^2}{\bar{p}_1} \bar{Q} \left[ 1 + \frac{1}{(\alpha + 1)} \left( p_2' \pi^{-(\beta+1)} - p_1' \right) \right] \]

Further assumptions
- Isentropic flow

Assumptions

- No viscous terms
- Quasi-1D
- Compactness
- Linear acoustics

Total Enthalpy

\[
\frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_2 \frac{u'_2}{\bar{c}_2} + (\gamma - 1) \frac{p'_2}{\gamma \bar{p}_2} + \frac{s'_2}{c_p} \right) = \frac{\dot{Q}'}{\bar{Q}} + \frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_1 \frac{u'_1}{\bar{c}_1} + (\gamma - 1) \frac{p'_1}{\gamma \bar{p}_1} + \frac{s'_1}{c_p} \right) - \frac{p'_1}{\gamma \bar{p}_1} - \frac{u'_1}{\bar{u}_1} + \frac{s_1}{c_p}
\]

Entropy

Mass

2nd law therm.

Further assumptions

- Isobaric combustion
- Low Mach number

Total Enthalpy

Entropy

Mass

2nd law therm.
Assumptions

- No viscous terms
- Quasi-1D
- Compactness
- Linear acoustics

Total Enthalpy

\[ \frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_2 \frac{u_2'}{\bar{c}_2} + (\gamma - 1) \frac{p_2'}{\gamma \bar{p}_2} + \frac{s_2'}{c_p} \right) = \frac{\dot{Q}'}{\dot{Q}} + \]

Entropy

\[ \frac{\lambda}{(\lambda \chi_1 - \chi_2)} \left( (\gamma - 1) \bar{M}_1 \frac{u_1'}{\bar{c}_1} + (\gamma - 1) \frac{p_1'}{\gamma \bar{p}_1} + \frac{s_1'}{c_p} \right) - \frac{p_1'}{\gamma \bar{p}_1} - \frac{u_1'}{\bar{u}_1} + \frac{s_1}{c_p} \]

Mass

2nd law therm.

Further assumptions

- Isobaric combustion
- Low Mach number

Total Enthalpy

\[ p_2' = p_1' \]

Entropy

\[ A_2 u_2' - A_1 u_1' = \frac{(\gamma - 1)}{\gamma \bar{p}} \dot{Q}' \]

Mass

2nd law therm.

(Non-Isentropic flow)
**OUTLINE**

**WHITE BOX**

\( Q' \rho' u' p' s' \)

Model Acoustic and entropy waves

\[
\begin{align*}
\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) &= 0 \\
\frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) &= -A \frac{\partial p}{\partial x} \\
\frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) &= A \frac{T}{T} \dot{q} \\
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) &= A q - A \frac{\partial p}{\partial t}
\end{align*}
\]

\[
\begin{align*}
 f &= B^+ e^{-i \omega x / c (1 + M)} = \frac{1}{2} \left( \frac{\bar{p}}{\rho c} + \bar{u} \right) \\
g &= B^- e^{i \omega x / c (1 - M)} = \frac{1}{2} \left( \frac{\bar{p}}{\rho c} - \bar{u} \right) \\
\bar{Q} &= \bar{p}_1 \bar{u}_1 A_1 c_p \bar{T}_1 \left( \frac{T_2}{T_1} - 1 \right) = \bar{u}_1 A_1 \frac{\gamma \bar{p}_1}{(\gamma - 1)} \left( \frac{T_2}{T_1} - 1 \right)
\end{align*}
\]

**CONNEXIONS**

\[
\begin{bmatrix}
1 & -R_{in} & 0 & 0 \\
0 & 0 & -R_{out} & 1 \\
T_{11} & T_{12} & -1 & 0 \\
T_{21} & T_{22} & 0 & -1
\end{bmatrix}_M \begin{bmatrix}
f_0 \\
g_0 \\
f_3 \\
g_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Gather all elements in a single matrix and compute acoustic response of the ensemble.

**STABILITY ANALYSIS.**

Study stability of the system

\[
\text{Growth rate}
\]

\[
\text{Unstable Region}
\]

Freq. - Neg. - Pos.
Definition of acoustic waves

1D Convective acoustic Wave equation

\[ \frac{D^2 p'}{Dt^2} - \bar{c}^2 \frac{\partial^2 p'}{\partial x^2} = 0 \]

Solution is the sum of two functions \( f \) and \( g \)
Definition of acoustic waves

1D Convective acoustic Wave equation

\[ \frac{D^2 p'}{Dt^2} - \bar{c}^2 \frac{\partial^2 p'}{\partial x^2} = 0 \]

Solution is the sum of two functions \( f \) and \( g \)

\( f \) and \( g \) are recognized as Riemann invariants
Definition of acoustic waves

1D Convective acoustic Wave equation

\[
\frac{D^2 p'}{Dt^2} - \bar{c}^2 \frac{\partial^2 p'}{\partial x^2} = 0
\]

Solution is the sum of two functions \( f \) and \( g \)

\( f \) and \( g \) are recognized as Riemann invariants

Knowing that harmonic oscillations are defined as

\[
()' = \hat{()} e^{i\omega t} \text{ and } \hat{()} = B e^{i\phi}
\]
Definition of acoustic waves

1D Convective acoustic
Wave equation

\[
\frac{D^2 p'}{Dt^2} - \bar{c}^2 \frac{\partial^2 p'}{\partial x^2} = 0
\]

Solution is the sum of two functions \( f \) and \( g \)

\( f \) and \( g \) are recognized as Riemann invariants

Knowing that harmonic oscillations are defined as

\[
()' = \hat{\phi} e^{i \omega t} \text{ and } \hat{\phi} = B e^{i \phi}
\]

\( f \) and \( g \) can be defined as

\[
f = B^+ e^{-i \omega x / \bar{c}(1 + M)} = \frac{1}{2} \left( \frac{\hat{p}}{\bar{\rho} \bar{c}} + \hat{u} \right)
\]

and

\[
g = B^- e^{i \omega x / \bar{c}(1 - M)} = \frac{1}{2} \left( \frac{\hat{p}}{\bar{\rho} \bar{c}} - \hat{u} \right)
\]
Definition of acoustic waves

1D Convective acoustic Wave equation

\[
\frac{D^2p'}{Dt^2} - \bar{c}^2 \frac{\partial^2 p'}{\partial x^2} = 0
\]

Solution is the sum of two functions \( f \) and \( g \)

\( f \) and \( g \) are recognized as Riemann invariants

Knowing that harmonic oscillations are defined as

\( (\hat{\phi}') = \hat{\phi} e^{i\omega t} \) and \( \hat{\phi} = Be^{i\phi} \)

\( f \) and \( g \) can be defined as

\[
f = B^+ e^{-i\omega x/\bar{c}(1+M)} = \frac{1}{2} \left( \frac{\hat{p}}{\hat{\rho} \bar{c}} + \hat{u} \right)
\]

and

\[
g = B^- e^{i\omega x/\bar{c}(1-M)} = \frac{1}{2} \left( \frac{\hat{p}}{\hat{\rho} \bar{c}} - \hat{u} \right)
\]

Downstream Travelling wave

Upstream Travelling wave
OUTLINE

WHITE BOX

\[ \dot{Q'}, \dot{p'}, \dot{u'}, \dot{p'}, \dot{s'} \]

Model Acoustic and entropy waves

\[ \begin{align*}
\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) &= 0 \\
\frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) &= -A \frac{\partial p}{\partial x} \\
\frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) &= \frac{A}{T} \dot{q} \\
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) &= A \dot{q} - A \frac{\partial p}{\partial t}
\end{align*} \]

\[ f = B^+ e^{-i\omega / \bar{c}(1 + M)} = \frac{1}{2} \left( \frac{\bar{p}}{\rho \bar{c}} + \bar{u} \right) \]
\[ g = B^- e^{-i\omega / \bar{c}(1 - M)} = \frac{1}{2} \left( \frac{\bar{p}}{\rho \bar{c}} - \bar{u} \right) \]
\[ \tilde{Q} = \tilde{\rho} \tilde{u}_1 A_1 c_p \tilde{T}_1 \left( \frac{\tilde{T}_2}{\tilde{T}_1} - 1 \right) = \tilde{u}_1 A_1 \frac{\gamma \tilde{p}_1}{(\gamma - 1)} \left( \frac{\tilde{T}_2}{\tilde{T}_1} - 1 \right) \]

Quasi 1D Conservation Equations
Spatial integration and the compact assumption
Linearization
Further Assumptions

Definition of waves

Isentropic ducts

Boundary conditions
Modeling of Flame dynamics

CONNEXIONS

\[
\begin{bmatrix}
1 & -R_{in} & 0 & 0 \\
0 & 0 & -R_{out} & 1 \\
T_{11} & T_{12} & -1 & 0 \\
T_{21} & T_{22} & 0 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
f_0 \\
g_0 \\
f_3 \\
g_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Gather all elements in a single matrix and compute acoustic response of the ensemble.

STABILITY ANALYSIS.

Growth rate

Unstable region

Study stability of the system
Isentropic ducts

\[ f = B^+ e^{-i\omega x / \bar{c}(1+M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho \bar{c}} + \hat{u} \right) \]

and

\[ g = B^- e^{i\omega x / \bar{c}(1-M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho \bar{c}} - \hat{u} \right) \]

Therefore

\[ f_2 = f_1 e^{-i\omega (x_2 - x_1) / \bar{c}(1+M)} \]

\[ g_2 = g_1 e^{i\omega (x_2 - x_1) / \bar{c}(1-M)} \]
OUTLINE

WHITE BOX

\[ \dot{Q}' \rho' u' p' s' \]

Model Acoustic and entropy waves

\[ \frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) = 0 \]
\[ \frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) = -A \frac{\partial p}{\partial x} \]
\[ \frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (p u s A) = A \frac{\partial q}{\partial x} \]
\[ \frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (p u h_t A) = Aq - A \frac{\partial p}{\partial t} \]

\[ f = B^+ e^{-i\omega x/c(1+M)} = \frac{1}{2} \left( \frac{\dot{p}}{\dot{\rho c}} + \dot{\rho} \right) \]
\[ g = B^- e^{i\omega x/c(1-M)} = \frac{1}{2} \left( \frac{\dot{p}}{\dot{\rho c}} - \dot{\rho} \right) \]

\[ \dot{Q} = \dot{\rho}_1 \dot{u}_1 A_1 c_p T_1 \left( \frac{T_2}{T_1} - 1 \right) = \dot{u}_1 A_1 \frac{\gamma \dot{p}_1}{\gamma - 1} \left( \frac{T_2}{T_1} - 1 \right) \]

✓ Quasi 1D Conservation Equations
✓ Spatial integration and the compact assumption
✓ Linearization
✓ Further Assumptions

Boundary conditions
Modeling of Flame dynamics

CONNEXIONS

\[
\begin{bmatrix}
1 & -R_{in} & 0 & 0 \\
0 & 0 & -R_{out} & 1 \\
T_{11} & T_{12} & -1 & 0 \\
T_{21} & T_{22} & 0 & -1
\end{bmatrix} \begin{bmatrix} f_0 \\
g_0 \\
f_3 \\
g_3 \end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0 \\
0 \end{bmatrix}
\]

Gather all elements in a single matrix and compute acoustic response of the ensemble.

STABILITY ANALYSIS.

Study stability of the system
Boundary Conditions

Acoustic flux through a boundary is given by

$$\dot{m}' h'_t = \left( \frac{p'}{c^2} \bar{u} + \bar{\rho} \bar{u}' \right) \left( \bar{u} u' + \frac{p'}{\bar{\rho}} \right)$$
Boundary Conditions

Acoustic flux through a boundary is given by

\[ \dot{m}' h'_t = \left( \frac{p'}{c^2} \bar{u} + \bar{\rho} u' \right) \left( \bar{u} u' + \frac{p'}{\bar{\rho}} \right) \]

If acoustic energy it is not dissipated through the boundaries then

\[ \dot{m}' h'_t = 0 \]
Boundary Conditions

Acoustic flux through a boundary is given by

\[
\dot{m}' h'_t = \left( \frac{p'}{c^2} \bar{u} + \bar{\rho} u' \right) \left( \bar{u} u' + \frac{p'}{\bar{\rho}} \right)
\]

If acoustic energy it is not dissipated through the boundaries then

\[
\dot{m}' h'_t = 0
\]

Inlet

- Open inlet: $h'_t = 0$
- Closed inlet: $\dot{m}' = 0$
- $f \begin{cases} \text{Open inlet} & h'_t = 0 \\ \text{Closed inlet} & \dot{m}' = 0 \end{cases}$
- $g \begin{cases} \text{Inlet} & f(1 + \tilde{M}) + g(1 - \tilde{M}) = 0 \\ & f(1 + \tilde{M}) - g(1 - \tilde{M}) = 0 \end{cases}$
Boundary Conditions

Acoustic flux through a boundary is given by

$$\dot{m'} h'_t = \left( \frac{p'}{c^2} \bar{u} + \bar{\rho} u' \right) \left( \bar{\rho} u' + \frac{p'}{\bar{\rho}} \right)$$

If acoustic energy is not dissipated through the boundaries then

$$\dot{m'} h'_t = 0$$

Open inlet/outlet

$$h'_t = 0 \quad f(1 + \bar{M}) + g(1 - \bar{M}) = 0$$

Closed inlet/outlet

$$\dot{m'} = 0 \quad f(1 + \bar{M}) - g(1 - \bar{M}) = 0$$
Boundary Conditions

Acoustic flux through a boundary is given by

$$
\dot{m}' h'_t = \left( \frac{p'}{c^2} \bar{u} + \bar{\rho} u' \right) \left( \bar{u} u' + \frac{p'}{\bar{\rho}} \right)
$$

If acoustic energy it is not dissipated through the boundaries then

$$
\dot{m}' h'_t = 0
$$

If acoustic energy it is no acoustic energy enters the system then

Inlet

$$
R_{in} = \frac{f}{g}
$$

Close end

$$
R_{in} = \frac{1 - \bar{M}}{(1 + \bar{M})}
$$

Open end

$$
R_{in} = -\frac{(1 - \bar{M})}{(1 + \bar{M})}
$$

Outlet

$$
R_{out} = \frac{g}{f}
$$

$$
R_{out} = \frac{1 + \bar{M}}{(1 - \bar{M})}
$$

$$
R_{out} = -\frac{(1 + \bar{M})}{(1 - \bar{M})}
$$
OUTLINE

WHITE BOX

\( \dot{Q}' \rho' u' p' s' \)

Model Acoustic and entropy waves

\[ \begin{align*}
\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) &= 0 \\
\frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) &= -A \frac{\partial p}{\partial x} \\
\frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) &= A \frac{\partial q}{\partial x} \\
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) &= A q - A \frac{\partial p}{\partial t}
\end{align*} \]

\[ \begin{align*}
f &= B^+ e^{-i \omega x / (1+M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho c} + \dot{u} \right) \\
g &= B^- e^{i \omega x / (1-M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho c} - \dot{u} \right)
\end{align*} \]

\[ \ddot{Q} = \ddot{p}_1 \bar{u}_1 A_1 c_p T_1 \left( \frac{T_2}{T_1} - 1 \right) = \bar{u}_1 A_1 \frac{\gamma \ddot{p}_1}{(\gamma - 1)} \left( \frac{T_2}{T_1} - 1 \right) \]

STABILITY ANALYSIS.

Gather all elements in a single matrix and compute acoustic response of the ensemble.

Gather all elements in a single matrix and compute acoustic response of the ensemble.

Study stability of the system

CONNEXIONS

\[ M = \begin{bmatrix}
1 & -R_{in} & 0 & 0 \\
0 & 0 & -R_{out} & 1 \\
T_{11} & T_{12} & -1 & 0 \\
T_{21} & T_{22} & 0 & -1
\end{bmatrix} \quad \begin{bmatrix}
f_0 \\
g_0 \\
f_3 \\
g_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} \]

Modeling of Flame dynamics

Quasi 1D Conservation Equations

Spatial integration and the compact assumption

Linearization

Further Assumptions

Definition of waves

Isentropic ducts

Boundary conditions
Modeling Flame Dynamics

\[ \frac{\dot{Q}}{\bar{Q}} = \frac{\hat{u}_1}{\bar{u}_1} \] where \[ \mathcal{F}(\omega) = G(\omega) e^{i\varphi(\omega)} \]

CFD or Experiments
OUTLINE

WHITE BOX
\[ \dot{Q}' \quad \rho' \quad u' \quad p' \quad s' \]
Model Acoustic and entropy waves

\[
\frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) = 0 \\
\frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) = -A \frac{\partial p}{\partial x} \\
\frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) = A \frac{\partial q}{\partial t} \\
\frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A q - A \frac{\partial p}{\partial t}
\]

\[
f = B e^{-i \omega x / (1 + M)} = \frac{1}{2} \left( \frac{\hat{p}}{\hat{c}} + \hat{u} \right) \\
g = B e^{i \omega x / (1 - M)} = \frac{1}{2} \left( \frac{\hat{p}}{\hat{c}} - \hat{u} \right)
\]

\[
\bar{Q} = \bar{p}_1 \bar{u}_1 A_1 c_p T_1 \left( \frac{T_2}{T_1} - 1 \right) = \bar{u}_1 A_1 \frac{\gamma \bar{p}_1}{\gamma - 1} \left( \frac{T_2}{T_1} - 1 \right)
\]

CONNECTIONS
\[
\begin{bmatrix}
1 & -R_{in} & 0 & 0 \\
0 & 0 & -R_{out} & 1 \\
T_{11} & T_{12} & -1 & 0 \\
T_{21} & T_{22} & 0 & -1
\end{bmatrix}
\begin{bmatrix}
f_0 \\
g_0 \\
f_3 \\
g_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
Gather all elements in a single matrix and compute acoustic response of the ensemble.

STABILITY ANALYSIS.
Study stability of the system

✓ Quasi 1D Conservation Equations
✓ Spatial integration and the compact assumption
✓ Linearization
✓ Further Assumptions

✓ Definition of waves
✓ Isentropic ducts
✓ Boundary conditions
✓ Modeling of Flame dynamics
In this case, the Mach number is considered zero.


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Network Models

Boundary Condition: Closed Inlet

Cross section change

Duct (cold)

Cross section change + Flame

Duct (hot)

Boundary Condition: Open outlet
In this case, the Mach number is considered zero.

\[ R_{\text{out}} = \frac{g_5}{f_5} \]

\[ R_{\text{in}} = \frac{f_0}{g_0} \]
In this case, the Mach number is considered zero.

\[
\begin{align*}
f_5 &= f_4 e^{-i\omega(x_5-x_4)/\bar{c}} \\
g_5 &= g_4 e^{i\omega(x_5-x_4)/\bar{c}} \\
f_1 &= f_0 e^{-i\omega(x_1-x_0)/\bar{c}} \\
g_1 &= g_0 e^{i\omega(x_1-x_0)/\bar{c}}
\end{align*}
\]
In this case, the Mach number is considered zero.

**Duct (hot)**

\[
\begin{bmatrix}
    f_5 \\
    g_5
\end{bmatrix} = \begin{bmatrix}
    e^{-i\omega(x_5-x_4)/\bar{c}} & 0 \\
    0 & e^{i\omega(x_5-x_4)/\bar{c}}
\end{bmatrix} \begin{bmatrix}
    f_4 \\
    g_4
\end{bmatrix}
\]

**Duct (cold)**

\[
\begin{bmatrix}
    f_1 \\
    g_1
\end{bmatrix} = \begin{bmatrix}
    e^{-i\omega(x_1-x_0)/\bar{c}} & 0 \\
    0 & e^{i\omega(x_1-x_0)/\bar{c}}
\end{bmatrix} \begin{bmatrix}
    f_0 \\
    g_0
\end{bmatrix}
\]
In this case, the Mach number is considered zero.

\[
(f_4 + g_4) = (f_3 + g_3) \xi \\
(f_4 - g_4) = (f_3 - g_3) \alpha [1 + \theta \mathcal{F}(\omega)]
\]
In this case, the Mach number is considered zero.

\[
\begin{bmatrix}
    f_4 \\
    g_4
\end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
    \xi + \alpha + \alpha \theta F(\omega) & \xi - \alpha - \alpha \theta F(\omega) \\
    \xi - \alpha - \alpha \theta F(\omega) & \xi + \alpha + \alpha \theta F(\omega)
\end{bmatrix}
\begin{bmatrix}
    f_3 \\
    g_3
\end{bmatrix}
\]
In this case, the Mach number is considered zero.

**Network Models**

**Boundary Condition: Open outlet**

**Duct (hot)**

**Cross section change + Flame**

**Duct (cold)**

**Cross section change**

**Duct (cold)**

**Boundary Condition: Closed Inlet**

\[
\begin{align*}
[f_5] &= D_3 [f_4] \\
[g_5] &= D_4 [g_4] \\
[f_4] &= F [f_3] \\
[g_4] &= F [g_3] \\
[f_3] &= D_2 [f_2] \\
[g_3] &= D_2 [g_2] \\
[f_2] &= C [f_1] \\
[g_2] &= C [g_1] \\
[f_1] &= D_1 [f_0] \\
[g_1] &= D_1 [g_0]
\end{align*}
\]
In this case, the Mach number is considered zero.

\[
\begin{bmatrix}
  f_5 \\
  g_5
\end{bmatrix} = T \begin{bmatrix}
  f_0 \\
  g_0
\end{bmatrix}
\]

where \( T = D_3 F D_2 C D_1 \)
In this case, the Mach number is considered zero.

\[
\begin{bmatrix}
  f_5 \\
  g_5
\end{bmatrix} = T
\begin{bmatrix}
  f_0 \\
  g_0
\end{bmatrix} \quad \text{where} \quad T = D_3 F D_2 C D_1
\]

Final matrix

\[
\begin{bmatrix}
  1 & -R_{\text{in}} & 0 & 0 \\
  0 & 0 & -R_{\text{out}} & 1 \\
  T_{11} & T_{12} & -1 & 0 \\
  T_{21} & T_{22} & 0 & -1
\end{bmatrix}
\begin{bmatrix}
  f_0 \\
  g_0 \\
  f_5 \\
  g_5
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix}
\]
In this case, the Mach number is considered zero.

\[
\begin{bmatrix}
  f_5 \\
g_5
\end{bmatrix} = T
\begin{bmatrix}
f_0 \\
g_0
\end{bmatrix} \quad \text{where} \quad T = D_3 F D_2 C D_1
\]

Final matrix

\[
\begin{bmatrix}
  1 & -R_{\text{in}} & 0 & 0 \\
  0 & 0 & -R_{\text{out}} & 1 \\
  T_{11} & T_{12} & -1 & 0 \\
  T_{21} & T_{22} & 0 & -1
\end{bmatrix}
\begin{bmatrix}
f_0 \\
g_0 \\
f_5 \\
g_5
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Solution comes by solving the characteristic equation

\[
\det(M) = 0 \quad \Rightarrow \quad T_{22} - R_{\text{out}} T_{12} + R_{\text{in}} T_{21} - R_{\text{in}} R_{\text{out}} T_{11} = 0
\]
**OUTLINE**

**WHITE BOX**

\[ \dot{Q}' \rho' u' p' s' \]

Model Acoustic and entropy waves

\[ \frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) = 0 \]
\[ \frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) = -A \frac{\partial p}{\partial x} \]
\[ \frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) = \frac{A}{T} \dot{q} \]
\[ \frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A \dot{q} - A \frac{\partial p}{\partial t} \]

\[ f = B^+ e^{-i\omega x/(1+M)} = \frac{1}{2} \left( \frac{\hat{p}}{\rho c} + \hat{u} \right) \]
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\[ \ddot{Q} = \bar{p}_1 \bar{u}_1 A_1 c_p T_1 \left( \frac{T_2}{T_1} - 1 \right) = \bar{u}_1 A_1 \frac{\gamma \bar{p}_1}{(\gamma - 1)} \left( \frac{T_2}{T_1} - 1 \right) \]

\[ \begin{bmatrix} 1 & -R_{in} & 0 & 0 \\ 0 & 0 & -R_{out} & 1 \\ T_{11} & T_{12} & -1 & 0 \\ T_{21} & T_{22} & 0 & -1 \end{bmatrix} M = \begin{bmatrix} f_0 \\ g_0 \\ f_3 \\ g_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

Gather all elements in a single matrix and compute acoustic response of the ensemble.

**CONNEXIONS**

**STABILITY ANALYSIS.**

Study stability of the system

- Quasi 1D Conservation Equations
- Spatial integration and the compact assumption
- Linearization
- Further Assumptions
- Definition of waves
- Isentropic ducts
- Boundary conditions
- Modeling of Flame dynamics
Stability analysis

Solving the characteristic equation

\[ T_{22} - R_{\text{out}} T_{12} + R_{\text{in}} T_{21} - R_{\text{in}} R_{\text{out}} T_{11} = 0 \]

we obtain a value for \( \omega \) (complex number)

\[ \omega = \omega_r + i \omega_i \]

Resonance frequency \hspace{1cm} Growth rate \hspace{2cm} (Stable or unstable)
Stability analysis

Solving the characteristic equation

\[ T_{22} - R_{\text{out}} T_{12} + R_{\text{in}} T_{21} - R_{\text{in}} R_{\text{out}} T_{11} = 0 \]

we obtain a value for \( \omega \) (complex number)

\[ \omega = \omega_r + i\omega_i \]

Resonance frequency \hspace{1cm} Growth rate

(Stable or unstable)

Freq. \hspace{1cm} Growth rate

Unstable Region

Neg. \hspace{1cm} Pos.
Stability analysis

\[ \omega = \omega_r - i\omega_i \]

Resonance frequency \quad Growth rate

Harmonic Oscillations a fluctuating quantity is expressed as

\[ a' = \hat{a} e^{i\omega t} \quad \Rightarrow \quad a' = \hat{a} e^{-\omega_i t} e^{i\omega_r t}. \]

Therefore

\[ \omega_i > 0 \quad \text{stability} \]
\[ \omega_i < 0 \quad \text{instability} \]
OUTLINE

WHITE BOX

\[ \dot{Q}' \quad \rho' \quad u' \quad p' \quad s' \]

Model Acoustic and entropy waves

\[ \frac{\partial}{\partial t} (\rho A) + \frac{\partial}{\partial x} (\rho u A) = 0 \]

\[ \frac{\partial}{\partial t} (\rho u A) + \frac{\partial}{\partial x} (\rho u^2 A) = -A \frac{\partial p}{\partial x} \]

\[ \frac{\partial}{\partial t} (\rho s A) + \frac{\partial}{\partial x} (\rho u s A) = A \dot{q} \]

\[ \frac{\partial}{\partial t} (\rho h_t A) + \frac{\partial}{\partial x} (\rho u h_t A) = A q - A \frac{\partial p}{\partial t} \]

\[ f = B^+ e^{-i \omega x / (1 + M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho c} + \dot{u} \right) \]

\[ g = B^- e^{-i \omega x / (1 - M)} = \frac{1}{2} \left( \frac{\dot{p}}{\rho c} - \dot{u} \right) \]

\[ \ddot{Q} = \bar{p}_1 \bar{u}_1 A_1 c_p \bar{T}_1 \left( \frac{\bar{T}_2}{\bar{T}_1} - 1 \right) = \bar{u}_1 A_1 \frac{\gamma \bar{p}_1}{(\gamma - 1)} \left( \frac{\bar{T}_2}{\bar{T}_1} - 1 \right) \]

CONNEXSIONS

\[ \begin{bmatrix} 1 & -R_{in} & 0 & 0 \\ 0 & 0 & -R_{out} & 1 \\ T_{11} & T_{12} & -1 & 0 \\ T_{21} & T_{22} & 0 & -1 \end{bmatrix} \]

\[ \begin{bmatrix} f_0 \\ g_0 \\ f_3 \\ g_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

Gather all elements in a single matrix and compute acoustic response of the ensemble.

STABILITY ANALYSIS.

Study stability of the system
The End