Dynamical Systems Approaches to Combustion Instability

Prof. R. I. Sujith
Indian Institute of Technology Madras
India

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What is dynamical systems theory?
Dynamical systems theory describes changes in systems that evolve in time
Image from Wikipidea
en.wikipedia.org/wiki/Buckling
What is a bifurcation?

A small smooth change in parameter values causes a sudden 'qualitative' change in behaviour.
Acknowledgement: Prof. Zinn’s notes, with permiss

Linear Stability/Instability

A system is linearly stable/unstable if every/any small amplitude disturbance decays/amplifies with time.

\[ p = \bar{p} + p' \quad p' \sim e^{\alpha t + iw t} \]

Stable operation

Criteria:
- Instability spontaneous
- Growth rate exponential (i.e., \( p' \sim e^{\alpha t} \); \( \alpha \sim \) growth rate)
- One of the system's natural modes is excited.

Note: \( \alpha \sim [G] - [L] \)
A system is nonlinearly unstable if some finite amplitude disturbance grows with time.

For triggering instability, the initial amplitude should be greater than a “threshold amplitude.”

From Prof. Zinn’s notes, with permission.
Two distinct kinds of instability can be identified – called Hopf bifurcation in dynamical systems theory.

![Super-critical and Sub-critical Bifurcation](image)

Regions of linear & nonlinear instability are different in sub-critical bifurcation / triggered instability.
Even if triggering does not occur, a sub-critical bifurcation is more dangerous for our system.
We discuss the application of tools from dynamical systems theory in thermoacoustics

Slow flow equations

$$\frac{dU}{dt} = \left(B_1 \varepsilon^2 + iB_2\right) U + \left(B_3 + iB_4\right)|U|^2 U$$

Numerical continuation to obtain bifurcation plots

Nonlinear time series analysis
Slow flow equations
A horizontal Rijke tube is modeled

We sidestep the effects of natural convection on the mean flow

Balasubramanian & Sujith (2008)
The acoustic field within the thermoacoustic system evolves as

\[
\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0
\]

\[
\frac{\partial p'}{\partial t} + \zeta p' + \gamma M \frac{\partial u'}{\partial x} = (\gamma - 1) \dot{q}_f(t) \delta(x - x_f)
\]

\[
u' = \sum_{j=1}^{N} U_j \cos(j \pi x) \quad \text{and} \quad p' = -\sum_{j=1}^{N} \frac{\gamma M}{j \pi} P_j \sin(j \pi x)
\]

Zinn & students (late 60s, early seventies)
Method of multiple scales can be used to obtain an amplitude equation

\[ \ddot{U} + a_0 \dot{U} + a_1 U + a_{2p} \left[ \sqrt{1 + 3 \cos(\pi x_f)} U(t - \tau) \right] - 1 = 0 \]

\[ U(t) = \varepsilon y(t) = A_1(t) \sin(t + \phi_1(t)) \]

Define multiple scales: \( t_0(t) = t \quad t_1(t) = \varepsilon t \quad t_2(t) = \varepsilon^2 t \)

\[ y(t) = Y_0(t_0, t_1, t_2) + \varepsilon Y_1(t_0, t_1, t_2) + \varepsilon^2 Y_2(t_0, t_1, t_2) + O(\varepsilon^3) + \ldots \]
Evolution equation for the slow flow is of the Stuart-Landau form

\[
\frac{dW}{dt} = \left( B_1 \varepsilon^2 + iB_3 \right) \left( \sigma - \sigma_H \right) W + \left( B_2 + iB_4 \right) |W|^2 W
\]

Slow flow equations for amplitude & phase:

\[
\frac{dA}{dt} = B_1 \left( \sigma - \sigma_H \right) A + B_2 A^3 ; \quad \frac{d\phi}{dt} = B_3 \left( \sigma - \sigma_H \right) + B_4 \phi^2
\]

Triggering amplitude:

\[ A \propto \left( \sigma - \sigma_H \right)^{1/2} \]
Numerical Continuation

Jahanke & Culick (1994)
Numerical continuation tracks the solution of a set of parameterized nonlinear equations

$$\frac{du}{dt} + F(u, \lambda) = 0$$

Simulations in time domain are replaced by iterative root finding of the corresponding constrained set of equations
The flow giving rise to a limit cycle can be also viewed as a map

Flow: \[ \frac{du}{dt} + F(u, \lambda) = 0 \]

Map: \[ u_{(n+1)T} = \Phi u_{nT} \]

\[ \Phi = \text{State Transition Matrix} \]

Floquet multipliers are the eigenvalues of the state transition matrix.
Increase in power destabilizes the system through a sub-critical Hopf bifurcation.

Subramanian et al. (IJSCD, 2010)
Stability variation with heater location is not monotonic

Subramanian et al. (IJSCD, 2010)
Nonlinear time series analysis
\[ \frac{d\vec{\chi}}{dt} = f(\vec{\chi}) \]

\[ \vec{\chi} = [\chi_1, \chi_2, \chi_3, \ldots \chi_n] \]
\[ \tilde{\chi} = [\chi_1, \chi_2, \chi_3, \ldots \chi_n] \]

In a CFD simulation, we calculate all the state variables;

In an experiment, we have one pressure transducer!
The phase space is reconstructed using embedding theorem.
The onset of instability is classified into soft and hard excitation.
Experimental data in a combustor with a bluff body flame holder - Nair & Sujith (2012)

Zinn & Lieuwen (2005) (p19): “Although large-amplitude disturbances are generally required to initiate unstable oscillations in non-linearly unstable systems, a system may be non-linearly unstable at low-amplitude disturbances that are of the order of the background noise level. This scenario is somewhat analogous to the hydrodynamic stability of a laminar Poiseuille flow.”
We investigate the role of noise in a ducted non-premixed flame system.
Oxygen plenum
Flush Nitrogen
Oxygen supply
Brass tube
Sub woofer
Fuel plenum

Quartz tube
Traverse mechanism
Methane-Nitrogen mixture

Jagadesan & Sujith (2012 symposium)
System undergoes transition via subcritical Hopf bifurcation.
Triggering instability is observed when the system is in hysteretic region.

Bifurcation diagram is separated into globally stable, globally unstable and bistable regions.
Spurts in amplitude are observed
System undergoes transition analogous to bypass transition to turbulence

Jagadesan & Sujith (Combustion Symposium 2012)
In thermoacoustics, the final state is “believed” to be a limit cycle, when driving balanced damping.
We observed intermittency in a turbulent swirl stabilized burner

Thampi & Sujith (2012)
Experimental setup: Ducted laminar premixed flame

Lipika Kabiraj

Kabiraj & Sujith (JFM 2012)
Limit cycle breaks leading to a state of flame detachment and reattachment
We see a subcritical Hopf bifurcation.
Bifurcation: limit cycle, quasi-periodic and intermittent oscillation

\[ x_f = 56.5 \text{ cm} \]

\[ x_f = 62 \text{ cm} \]

\[ x_f = 64 \text{ cm} \]
During limit cycle, wrinkles originate at the base of the flame and propagate downstream.
Flame response during quasi-periodic oscillations shows elongation, neck formation, cusping & pinch off.
Dynamical behavior before flame blowout: Intermittent oscillations

\[ x_f = 64 \text{ cm to } x_f = 68.5 \text{ cm} \]
Two types of bursts are observed along with sections of laminar state

L- laminar state,  B1 & B2- two types of burst,  
F-fixed point (no oscillation)  

$x_f = 64 \text{ cm}$
Recurrence plot is a graphical representation for visualizing system dynamics from short time series.

Embedding theorem

A square binary recurrence matrix

\[
R_{ij} |_{N \times N}
\]

\[
R_{ij} = \Theta(\epsilon - ||\vec{x}_i - \vec{x}_j||), \quad i, j = 1, \ldots, N
\]

\[
p(t) - \text{Time series}
\]

\[
p(t) \quad p(t + 2\tau) \quad p(t + \tau)
\]

Reconstructed phase space
Recurrence plot: limit cycle and quasi-periodic oscillation

\( a \)

\[ \begin{array}{c}
\text{Time} \\
1000 & 1300 & 1500 & 1700 \\
\end{array} \]

\( \Rightarrow T \)

\( b \)

\[ \begin{array}{c}
\text{Time} \\
3200 & 3300 & 3400 & 3500 \\
\end{array} \]
Features of RP for type-II intermittency
Flame attachment and reattachment leads to intermittent oscillations
Simultaneous to these bursts, flame oscillates violently, lifts off & then reattaches to the burner rim.

a-c: Laminar state (L), d-o: Burst state (B2), p-r: Reattachment.
What does dynamical systems theory tell us about instabilities in a turbulent combustor?
Thermoacoustic oscillations have a “micro-structure” which can be revealed using nonlinear dynamics calculation.
Can be used to detect the onset* of an impending instability before the instability occurs

*Patent pending
In summary, dynamical systems theory provides us with tools that can be used to gain new insights.

Analytical & numerical tools to study thermoacoustics using bifurcation theory.

Time series analysis can be used for studying thermoacoustic systems experimentally.

Thermoacoustic systems have much richer behavior than just limit cycles.